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Abstract

The paper presents a coalitional bargaining model with a peculiar type of partial breakdown: the Burning Coalition Bargaining Model. Rather than triggering the end of all negotiations or the exclusion of some players from the game, as already proposed in the literature, in this model the rejection of a proposal causes the possibility of the proposed coalition to vanish. Under this type of partial breakdown and adopting a standard rejecter-proposes protocol, 0-normalized, 3-players games are examined for extreme values of the breakdown probability. When such probability is equal to one, efficiency is more difficult to obtain than in models adopting discounting. Furthermore, when an efficient outcome is attained, the final pay-offs distribution reflects the strength of players in the game, with strength being defined by belonging to more valuable coalitions. The same feature is retained when considering a probability of breakdown approaching zero.

Keywords: Bargaining Theory; Bargaining protocols; Coalition formation; Efficiency; Partial breakdown.

J.E.L.: C71; C78.

1 Introduction

The seminal paper of Rubinstein (1982) has undoubtedly constituted a breakthrough in the field of bargaining theory. Looking back at the novelty introduced by this model, the cost of delaying the achievement of an agreement, it surprises that it has not been thought before, but this is often a typical characteristic of great innovations. Besides the ability of the model to incorporate a fundamental and natural economic concept such as the cost of delay, its shining fortune is due to the fact that this introduction allows to solve the model uniquely through a robust equilibrium concept: subgame perfect equilibrium (SPE). The other important reason contributing to its popularity is the support for the Nash bargaining solution (Nash Jr, 1950; Nash, 1953), that, in turn, can be considered as one of the most prominent solution concepts for cooperative games. With the introduction of the Rubinstein bargaining model, that research agenda going under the name of Nash program, whose objective is to find a non-cooperative support to cooperative solution concepts, has encountered its highest momentum (Binmore, 1986).

Once exhausted the analysis of two- and multi-players bargaining problems[1] by using the Rubinstein's model, the academic interest has shifted towards the more complex scenario of coalitional bargaining, considering mainly transferable utility (TU) games, but with some exceptions analysing non-transferable utility (NTU) ones (Rogna, 2019). In the two- and multi-players bargaining model of Rubinstein, the cost of delay has been given different interpretations, corresponding to different technical implementations. The most popular is discounting, according to which future pay-offs worth less than present ones. A second variant, still proposed in Rubinstein (1982), considers the act of negotiation as a costly activity. Finally, Binmore et al. (1986) has proposed a third way, with the cost of delay residing in the possibility of missing the opportunity to get the sum over which bargainers are negotiating. Technically, this last variant has been introduced in the form of an exogenous probability of breakdown of negotiations.

Whereas in two- and multi-players bargaining, except for minor differences – see, on this regard, Serrano (2004) –, these variants all lead to the same result, in coalitional bargaining this does not hold any more. Furthermore, whereas the interpretation and the implementation of discounting remains the same in the passage to coalitional models, this is not the case for the breakdown of negotiation[2]. van den Brink et al. (2013) adopts a model with risk of total breakdown of negotiations, but Hart and Mas-Colell (1992) and

[1]Along the paper it will be adopted the following terminological convention: multi-player bargaining indicates a bargaining situation entailing more than two players where only one coalition, namely, the coalition with all players can be formed; coalitional bargaining, instead, assumes both the presence of more than two players and the possibility to form coalitions that are subsets of the set of all players; two players bargaining is self-explanatory.

[2]The fixed cost of bargaining is not mentioned here since, to the best of my knowledge, I am not aware of coalitional bargaining models adopting this interpretation.

Hart and Mas-Colell (1996) prefer to rely on the idea of partial breakdown, where only some players suffer the probability of being excluded after the rejection of a proposal. This last variant is not a mere technical departure, but rather a different theoretical conceptualization of breakdown. This is particularly so when the player subject to the risk of exclusion is the proposer. In such case, in fact, the breakdown should be interpreted as a punishment of the other players towards the proposer, taking the form of an exclusion, for her inability to propose an acceptable allocation (Hart and Mas-Colell, 1996).

The present paper, instead, aims at reintroducing the original idea of Binmore et al. (1986) for which delay may cause an opportunity to vanish. It does so by presenting the Burning Coalition Bargaining Model (BCBM) whose peculiarity is the substitution of discounting and breakdown with the possibility of dissolution of the worth of a coalition if the proposed allocation is not accepted by its members. Although it could be defined as a variant of partial breakdown, we believe its departure is quite significant and worth to be analysed. In particular, it is very sharp the difference in results and in the equilibrium pay-offs compared to both the coalitional bargaining models based on discounting and the ones based on partial breakdown à la Hart and Mas-Colell (1996). The formers, in fact, generally reach a very egalitarian outcome, allocating an equal share of the grand coalition worth to all players whenever this allocation belongs to the core (Chatterjee et al., 1993; Okada, 1996; Yan, 2003). Hart and Mas-Colell (1996) and models alike, e.g. Krishna and Serrano (1995); van den Brink and Funaki (2010) and Calvo and Gutiérrez-López (2016), find support for the Shapley value and some variants of it. In the present model, even when the grand coalition is obtained, the distribution of pay-offs is far less egalitarian than models based on discounting, but still very different from the Shapley value. The bargaining strength of a player is not merely due to her position as proposer rather then responder, but it is also, and particularly, represented by the worth of the coalitions she belongs to compared to the one she does not belong to. This last aspect gets fully reflected in her final equilibrium pay-off for all possible values of the parameter determining the probability of partial breakdown.

2 Review of the Relevant Literature

As anticipated in the introduction, the extension of the Rubinstein's model to coalitional bargaining has not been straightforward. In reality, even the passage from two-players to multi-players has encountered some complications. In multi-players bargaining, adopting the fixed discount interpretation, both Sutton (1986) and Osborne and Rubinstein (1990) have recognized that there is a large multiplicity of equilibria. Actually, every efficient pay-off vector that grants to every player at least her outside option can be sustained in perfect equilibrium if players are enough patient. A first way to recover the uniqueness of the Nash result in multi-players bargaining – in the limit of the common discount factor approaching unity – has been put forth by Chae and Yang (1994) through a model where the final agreement is reached through a series of bilateral negotiations. Krishna and Serrano (1996) offer a second method to refine the mentioned abundance of equilibria, restoring the asymptotic support of the Nash bargaining solution as unique subgame perfect equilibrium (SPE). They achieve such result by adopting contingent offers, a mechanism rooted into the consistency principle of Lensberg (1988) for which unanimous acceptance of an agreement is not required since players can exit after reaching partial agreements. Miyakawa (2008) and Britz et al. (2010) prefer to adopt a model with risk of breakdown of negotiations rather then discounting. The result is similar to the previous ones with uniqueness being reached in stationary subgame perfect equilibrium (SSPE), although they find support, as the risk tends to vanish, for the asymmetric Nash bargaining solution (Kalai, 1977). In both cases asymmetries are generated by the adoption of a random proposer model with, in the former, asymmetric recognition probabilities and, in the latter, Markow recognition probabilities.

In coalitional bargaining, the inherently more complex structure of the game generates a higher variance of outcomes. This is further increased by the fact that the adaptation of the two players Rubinstein's model to such situation is more open to variations as well. The way in which the proposer is selected, both at the beginning of the game and after a potential rejection of a proposal, the nature of the offers, contingent or not, and the possibility – or impossibility – to continue the bargaining process after one coalition has successfully formed are examples of crucial elements that can influence the results (Rogna, 2019). As in multi-players bargaining, the multiplicity of equilibria remains a serious issue. In particular, Chatterjee et al. (1993) have shown that the restriction to subgame perfection is not helpful to sharpen results for values of the common discount factor sufficiently high since, in such case, every vector of pay-offs granting to each player at least her disagreement point pay-off can be sustained as an SPE equilibrium.

Stationarity is a first necessary step to reduce the number of feasible equilibria, but, alone, it does not necessarily lead to uniqueness. In fact, assuming the underlying coalitional game being balanced, Moldovanu and Winter (1995), Evans (1997) and Kim and Jeon (2009) show, under different model specifications, that every Core allocation can be sustained as an SSPE equilibrium. Whereas the first two works do not make use of discounting, the paper of Kim and Jeon (2009), that adopts the rejecter-proposes protocol, does. A major result of this paper is to show that the set of SSPE equilibria corresponds to the set of optimal solutions of a minimization program. By adopting discounting and the random proposer protocol, Yan (2003) obtains a unique SSPE equilibrium, according to which the proportion of the final pay-off of each player over the worth of the grand coalition will mirror her recognition probability. If the resulting allocation vector is not a Core element, delay and inefficiency will arise.

Given the result of Yan (2003), it comes with no surprise that several papers (e.g. Okada (1996), Compte and Jehiel (2010) and Okada (2011)), considering the case of equal recognition probabilities of players, obtain a strongly egalitarian equilibrium, namely, they all support the Egalitarian Solution (Thomson, 1983)[3]. However, when such redistributive SSPE allocation vector is not in the Core, inefficiencies arise as predicted in Yan (2003).

Once considering the risk of negotiations' breakdown, it must be noted that partial breakdown has prevailed in coalitional settings. In fact, the seminal papers introducing it, namely, Hart and Mas-Colell (1992), Krishna and Serrano (1995) and Hart and Mas-Colell (1996), all adopt a partial breakdown assumption. This implies that, when a proposal is rejected, some of the players face the risk of being excluded and, consequently, all the coalitions they were belonging to dissolve. Clearly, in two- or multi-players bargaining, this equates to a total breakdown of negotiations. The peculiarity of the three models is to find support for the Shapley value rather than for the Egalitarian Solution or other strongly redistributive Core allocations. Again, it must be noted that, in two- or multi-players bargaining, assuming players are bargaining in terms of utils, the Nash bargaining solution and the Shapley value coincide.

When the player to face the risk of exclusion is the proposer, Hart and Mas-Colell (1992) show that the Shapley value is supported, in expectation, as an SSPE for monotonic games. Krishna and Serrano (1995) refine this result proving that, for particular values of the parameter determining the probability of breakdown, the same equilibrium is obtained as an SPE. Hart and Mas-Colell (1996) extend the analysis to the case of NTU games, finding support for the consistent values of Maschler and Owen (1989), and, for the TU case, they test several variants among which the possibility that it is not the proposer to face the risk of exclusion, but the responders with equal probability. In such case, the Egalitarian Solution emerges as an SSPE. In the intermediate case, when both the proposer and the responders, these lasts with equal probability, face the possibility of exclusion, the SSPE result is a convex combination of the Shapley value and the Egalitarian Solution, named Egalitarian Shapley value. Further variants include the work of Calvo (2008), where all the remaining players in the game, not only responders, are equally likely to be excluded and that supports the Solidarity value; the one of van den Brink et al. (2013) that adds the possibility of a total, besides the one of partial, breakdown and that finds support for the Egalitarian Shapley value and the ones of van den Brink and Funaki (2010) and Calvo and Gutiérrez-López (2016), that, adding a common discount factor, obtain, in SSPE, the so called Discounted Shapley value. Kawamori (2016) shows that, for the underlying coalitional game respecting certain conditions, this result holds even when players are not

[3] Note that the Egalitarian Solution is also know as the Equal Split (Hart and Mas-Colell, 1996) or the Equal Division (van den Brink, 2007) allocation.

obliged to make a proposal to every player in the game[4], being, instead, allowed to propose only to the members of a specific coalition. Finally, Pérez-Castrillo and Wettstein (2001) show that, by substituting the random selection of the proposer with a bidding stage in the model of Hart and Mas-Colell (1992), the Shapley value is obtained not in expectation but as the unique equilibrium of the game.

3 The Burning Coalition Bargaining Model

3.1 Model's description

Although we will limit our analysis to 3-players game, we start by providing a general description of the game encompassing any number of players. The bargaining game we are going to consider can be fully described by a 4-tuple: $B = (N, v, \Sigma, \alpha)$. The first two elements of such tuple define the underlying coalitional game, with N being the set of players – $N = \{1, 2, ..., |N|\}$ – and $v : 2^{|N|} \to \mathbb{R}_+ \setminus \infty$ being the characteristic function that defines the worth of each possible subset of N. As usual, we have $v(\emptyset) = 0$. Note that we assumed the worth of each coalition being finite and non-negative. This last assumption, although not crucial, allows to define the set of coalitions in B as the power set of N – denoted as $\mathscr{P}$ – without having a further mapping function that defines the set of feasible coalitions. In fact, an eventual infeasible coalition can be assigned the worth of zero without altering the bargaining structure of the game.

The set Σ represents the whole strategic space of the game. Assuming, although we will see this is not the case, that the bargaining process is infinite, we have $\Sigma = \times_{t=1}^{\infty} \sigma_t$ with $\sigma_t = \times_{i=0}^{|N|} \sigma_{i,t}$. Note that $\sigma_{0,t}$ represents the move, at each time period t, done by the random mechanism operating in the game. The last element of the tuple is actually the parameter determining the probability of partial breakdown. Being this the definition of all elements of our model, let us see, concretely, how it works.

The model can be described as a standard Rubinstein-type sequential bargaining model with a peculiar risk of partial breakdown, where only pure strategies are allowed. With regard to the selection of the proposer, we adopt the rejecter-proposes rule á la Chatterjee et al. (1993). At the beginning of the game, the first move is reserved to the random mechanism that will select a full order of players. All of the $n!$ permutations[5] are equally likely. The first player in the selected order is the first proposer. Since there is no possibility to "pass" its own turn and given the rejecter-proposes rule, the selection of a permutation rather than of just the first proposer is useful only for having

[4]Note that such assumption is present both in Hart and Mas-Colell (1992) and Calvo and Gutiérrez-López (2016).

[5]We adopt the convention for which $n = |N|$.

a given order of responders once a proposal is made.[6] Once a player, let us say i, has been appointed as proposer, her strategic choice is given by a set of 2-tuples $(S, \boldsymbol{x}_S)$. The first element of the tuple indicates the coalition selected by i that must be an element of $\mathcal{S}_t$, the set of active coalitions at time t, and must be such that $S \ni i$. The vector $\boldsymbol{x}_S \in \mathbb{R}^s$ – with $s = |S|$ – has for elements the proposed allocation for each member of coalition S (including i) and must be such that $\sum_{j \in S} x_j \leq v(S), x_i \geq 0, \forall i \in S$. We then have:

$$\sigma^p_{i,t} = \{(S, \boldsymbol{x}_S)\}, \ \forall S \in \mathcal{S}_t, S \ni i, \ \sum_{j \in S} x_j \leq v(S), \quad i \text{ is a proposer.}$$

The superscript p is used to indicate the strategy of a proposer, whereas superscript r will be used to indicate the responders' strategy.

Responders have a simple dichotomous choice consisting in accepting or rejecting the proposal. They answer sequentially according to the order defined in the first stage of the game, starting with the player coming soon after the proposer. It must be noted that such order of answers is inconsequential and that the model would actually run identically if we were assuming that responses are given contemporaneously. Players not belonging to the coalition pointed by the proposer do not have any available action:

$$\begin{aligned} \sigma^r_{j,t} &= \{\text{accept, reject}\}, \qquad && \forall j \in S, j \neq i, \quad S \text{ being the proposed coalition.} \\ \sigma^1_{k,t} &= \emptyset, && \forall k \in N \setminus S. \end{aligned}$$

When unanimous acceptance occurs, the members of the proposed coalition exit the game with the agreed pay-offs: $\pi_i = x_i$ and $\pi_j = x_j, \forall j \in S$, whereas the remaining players can continue to bargain. The next proposer will be the player following i according to the original selected order from which the exited players have been simply removed.[7]

The novelty of this model, as mentioned, emerges after the eventual rejection of a proposal. In such case, in fact, the move will pass to the random mechanism that, with probability $\alpha \in (0, 1]$, will eliminate ("burn") the proposed coalition from the set of feasible coalitions, whereas, with probability $1 - \alpha$, the game will move to the next period remaining unchanged. Here, the rejecter-proposes rule applies and, therefore, the first of the rejecters will become the new proposer. Note therefore that, under the rejecter-proposes rule, the random mechanism operates only at the first stage of the game, by selecting the proposers' order, and, after a rejection, in order to determine if the proposed coalition will be burned or not.

[6] As in Chatterjee et al. (1993), Okada (1996), Compte and Jehiel (2010) and in most of coalitional bargaining models, the order of responders is inconsequential, therefore it could be left to the proposer to choose it. However, given its irrelevance, a predefined order seems the easiest solution.

[7] In our 3-players case with zero normalized characteristic function, the possibility of continuing to bargain is, however, inconsequential.

Finally, a last comment regarding the time horizon of the game. Clearly, having defined $\mathcal{S}_t$ as the set of coalitions still available at time t, we then have: $\mathcal{S}_1 \equiv \mathscr{P}$; $|\mathcal{S}_t| \leq |\mathscr{P}|, \forall t > 1$ and $|\mathcal{S}_{t\to\infty}| = |N|$. The last equality implies that, in a bargaining process approaching infinity, the set of available coalitions will end up being equal to the set of singleton coalitions. Therefore, also in the present model, as in Hart and Mas-Colell (1996), the parameter determining the risk of breakdown, α, can be seen as a substitute for discounting.

4 Analysis of 0-normalized, 3-players Games with $\alpha = 1$

We start the analysis of our bargaining model restricting the attention to a specific class of underlying coalitional games: 3-players, 0-normalized, games. Let us then formally define what a 0-normalized game is.

Definition 4.1. (*0-Normalized Game*). Given a coalitional game $\Gamma = (N, v)$, its 0-normalized form, $\Gamma_0 = (N, v_0)$[8], is given by:

$$v_0(S) = v(S) - \sum_{i \in S} v(\{i\}), \quad \forall S \subseteq N.$$

Further note that the non-negativity condition of characteristic values mentioned in the previous section holds even after the 0-normalization: $v_0(S) \geq 0, \forall S \subseteq N$. In our analysis, we will follow the convention of having $N = \{j, i, k\}$ as set of players, with j being considered the "strong", i the "middle" and k the "weak" player. These adjectives originate from the following assumption:

Assumption 4.1. Given a 0-Normalized coalitional game Γ, with player set $N = \{j, i, k\}$, its characteristic function will have the following property:

$$v(N) \geq v(i, j) \geq v(j, k) \geq v(i, k) \geq 0.$$

Under Assumption 4.1, forgetting the grand coalition (N), j is the most advantaged player, being member of the two coalitions with highest worth, k is weak being in the opposite situation, whereas i stays in a middle position since she belongs to the coalitions with highest and lowest worth.

Finally, although not all our results will be limited to the class of convex games, this particular class plays a central role in our examination of efficiency conditions.

Definition 4.2. (*Convex Game*). Given a coalitional game $\Gamma = (N, v)$, the game Γ is said to be convex if:

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall S, T \subseteq N.$$

[8]Note that, in the rest of the paper, since we will always assume 0-normalized games, the subscript *0* will be omitted.

Note that the condition for strict convexity is identical to the one of convexity except that the inequality must hold strictly.

We will firstly consider the limit case of $\alpha = 1$. Willing to make a parallel with the fixed discount model, the certain probability of the worth of a coalition to vanish after a rejection corresponds to $\delta = 0$ in such model, with δ being the discount factor. Under this condition, we would have a simple dictator game, whose unique SPE equilibrium would be for proposer q to choose $S : \max_{S \ni q, S \subseteq N} v(S)$ and to offer an infinitesimal amount ϵ to all other players in S, that will all accept. Clearly, in a convex game, each player would propose the grand coalition N. Convexity, therefore, is sufficient to guarantee an efficient bargaining process, either because the agreement is reached in the first step and because the coalition granting the highest overall welfare is chosen.

In the BCBM, instead, things run far less smoothly. This is due to the fact that, although a rejection would cause the proposed coalition to vanish, still the rejecter has other options. With $\alpha = 1$, the game has a finite horizon, therefore SPE can be individuated through backward induction. Figure 1 in Annexes A1 shows the extensive form representation of the game given the rejecter-proposes rule and assuming i being the first player in the randomly selected permutation. It can be observed that the game has the structure of a ramified centipede game. Let us now introduce a Lemma that will be helpful in determining the SPE of the game.

Lemma 4.1. *Given an underlying convex coalitional game Γ following Assumption 4.1, for k being the first proposer in a 3-players BCBM with $\alpha = 1$, k cannot make an acceptable proposal granting her a strictly positive pay-off by calling a 2-players coalition.*

Proof. The assumption of k choosing a 2-players coalition implies that k has two options: selecting $\{j, k\}$ or selecting $\{i, k\}$. Suppose she selects the former, implying k needs to make an offer x_j to j. By rejecting such proposal, j has two options, calling coalition $\{i, j\}$ or calling N.
Case 1: j calls $\{i, j\}$.
In such case, j must make a proposal x_i to i that further has two options in case of rejection, calling $\{i, k\}$ or calling N. The first option can be excluded immediately, since k, by rejecting the offer, would have the possibility to call N and, since no more multi-player coalitions are left in the game, she would act as a dictator, granting for herself the whole worth of N: $v(N)$. By calling N, instead, i would have the possibility to obtain a positive pay-off. In fact, no coalitions including j are present any more in the game, therefore 0 could be offered to j, whereas k must be offered $v(i, k)$, since she could in any case obtain this pay-off by rejecting the offer of i. Therefore, $x_i = v(N) - v(i, k)$. This implies that j, by calling $\{i, j\}$, could grant to herself $v(i, j) - [v(N) - v(i, k)] = v(j, k) + v(i, k) - v(N) \leq 0$, with the last inequality originating from convexity. In Case 1, player j does not have any possibility to obtain a positive pay-off.
Case 2: j calls N.

In such case, j must make an offer x_i and x_k to both players i and k. Player k has two options in case of rejection, calling $\{i,k\}$ or $\{i,j\}$. In both cases, i can grant to herself the whole value of the coalition, since neither k nor j would have any multi-player coalition including them still available after a rejection. By assumption, $v(i,j) \geq v(i,k)$, therefore $x_i = v(i,j)$ for i to accept. Player k, instead, has the sole possibility, by rejecting, to call $\{i,k\}$, but she will have to offer i at least $v(i,j)$, since this is the pay-off i is guaranteed by rejecting. By the same assumption, $v(i,j) \geq v(i,k)$, k does not have any possibility to receive a pay-off greater than zero, implying $x_k = 0$. This means that j, by rejecting k's proposal and calling N, can grant herself $v(N) - v(i,j)$.
This further implies that k, for making an acceptable proposal to j must give her a minimum of $v(N) - v(i,j)$, leaving for herself $v(j,k) - [v(N) - v(i,j)] = v(j,k) + v(i,j) - v(N) \leq 0$. It is easy to see that, by calling $\{i,k\}$ instead of $\{j,k\}$, k would run into a similar situation: $\pi_k = v(i,k) - [v(N) - v(i,j)] = v(i,k) + v(i,j) - v(N) \leq v(j,k) + v(i,j) - v(N) \leq 0$. No more cases are possible. ■

If we would like to generalize beyond a convex game the maximum pay-off that k can obtain by calling a 2-players coalition when $\alpha = 1$, it is immediate to see that $\pi_k = \max(v(j,k) + v(i,k) - v(N), 0)$. This is the obtainable pay-off by calling coalition $\{j,k\}$ under the mentioned assumption: $v(N) \geq v(i,j) \geq v(j,k) \geq v(i,k) \geq 0$. Clearly, under convexity, $\pi_k = 0$, thus the result of Lemma 4.1. If the weak player has no possibility to obtain a positive pay-off by calling a 2-players coalition when selected as first proposer in a convex BCBM with $\alpha = 1$, players j and i, instead, have such possibility upon certain conditions.

Lemma 4.2. *Given an underlying coalitional game Γ following Assumption 4.1, for j being the first proposer in a 3-players BCBM with $\alpha = 1$, she can call coalition $\{j,k\}$, make an acceptable proposal, and grant to herself:*

$$\pi_j = \begin{cases} v(j,k), & \text{if } v(N) + v(i,k) \leq 2v(i,j), \\ \max(v(j,k) + 2v(i,j) - v(N) - v(i,k), 0), & \text{otherwise.} \end{cases}$$

If the first proposer is i, she can call coalition $\{i,k\}$, make an acceptable proposal, and grant to herself:

$$\pi_i = \begin{cases} v(i,k), & \text{if } v(N) + v(j,k) \leq 2v(i,j), \\ \max(v(i,k) + 2v(i,j) - v(N) - v(j,k), 0), & \text{otherwise.} \end{cases}$$

Proof. The proof runs identically to the proof of Lemma 4.1, with the result obtained by simply applying backward induction. ■

Two aspects worth to be noticed in Lemma 4.2. First of all, players j and i have the possibility to get a positive pay-off by calling a 2-players coalition if, respectively, $v(N) + v(j,k) < 2v(i,j) + v(i,k)$ and $v(N)+v(i,k) < 2v(i,j)+v(j,k)$. Therefore, $\pi_j = v(j,k)+2v(i,j)-v(N)-v(i,k)$ *iff* $2v(i,j) < v(N) + v(i,k) < 2v(i,j) + v(j,k)$. If we substitute $v(j,k)$ with

$v(i,k)$ and vice-versa, we obtain the pay-off and the relative condition for player i. Further note that convexity is not mentioned in Lemma 4.2, implying these results are independent from such assumption.

The second interesting fact to notice is that both j and i obtain more by calling the coalition with the lowest worth among the 2-players coalitions they belong to. In particular, if the underlying coalitional game is convex, both players could not obtain any positive pay-off by calling coalition $\{i,j\}$. It is easy to see that, in such case, we would have $\pi_j = v(i,j) - [v(N) - (v(j,k) - v(i,k))] = v(i,j) + v(j,k) - v(N) - v(i,k) \leq 0$ and $\pi_i = v(i,j) - [v(N) - v(j,k)] = v(i,j) + v(j,k) - v(N) \leq 0$. Both the ending inequalities are originating from convexity: $v(N) \geq v(i,j) + v(j,k)$. In order to derive the conditions for efficiency in the BCBM with $\alpha = 1$, we need to state the pay-offs obtainable by the three players by calling the grand coalition in the first bargaining round.

Lemma 4.3. *Given an underlying coalitional game Γ following Assumption 4.1, when selected as first proposer in a BCBM with $\alpha = 1$, players k, j and i, by calling the grand coalition, obtain the following pay-offs:*

$$\pi_k = \max(v(N) + v(j,k) - 2v(i,j) - v(i,k), 0).$$

$$\pi_j = \begin{cases} \max(v(N) + v(j,k) - v(i,j) - v(i,k), 0), & \textit{if } v(j,k) + v(i,k) \leq v(i,j), \\ \max(v(N) - 2v(i,k), 0), & \textit{otherwise.} \end{cases}$$

$$\pi_i = \begin{cases} \max(v(N) - v(i,j), 0), & \textit{if } v(j,k) + v(i,k) \leq v(i,j), \\ \max(v(N) - v(j,k) - v(i,k), 0), & \textit{otherwise.} \end{cases}$$

Proof. Once again, the proof simply requires to use backward induction. ■

As in Lemma 4.2, the results of Lemma 4.3 are independent from convexity. However, when strict convexity holds, it is possible to observe that players j and i always enjoy a positive pay-off by calling N. Actually, convexity alone is sufficient to assure this result, provided that some inequalities in Assumption 4.1 hold strictly. For example, convexity and $v(N) \geq v(i,j) \geq v(j,k) \geq v(i,k) > 0$ are sufficient conditions for π_j and π_i in Lemma 4.3 to be always strictly positive. With regard to π_k in Lemma 4.3, instead, even strict convexity is neither a sufficient nor a necessary condition for positiveness. In fact, under strict convexity, we have $v(N) = v(i,j) + v(j,k) + \eta$, with $\eta > 0$. Therefore $v(N) + v(j,k) - 2v(i,j) - v(i,k) > 0 \Rightarrow v(i,j) + v(j,k) + \eta + v(j,k) - 2v(i,j) - v(i,k) > 0 \Rightarrow \eta > v(i,j) + v(i,k) - 2v(j,k)$. The last inequality, however, may fail to be true. Just consider, for example, a 0-normalized game where $v(N) = 15, v(i,j) = 10, v(j,k) = 4$ and $v(i,k) = 2$. Strict Convexity and Assumption 4.1 hold and we have $\eta = 1$. But then, $1 > 4$ is clearly false, thus implying that strict convexity is not a sufficient condition for π_k to be positive. That is not a necessary condition as well can be easily seen by considering that the RHS of the inequality $\eta > v(i,j) + v(i,k) - 2v(j,k)$

may be negative, thus not requiring η to be strictly positive.

We are now ready to state the conditions for efficiency in a 3-players, 0-normalized BCBM when $\alpha = 1$.

Proposition 4.1. *Given a strictly convex underlying coalitional game Γ following Assumption 4.1, whenever player j is selected as first proposer, the outcome will be efficient.*

Proof. From Lemmas 4.2 and 4.3, we have the maximal pay-offs j can obtain by calling, respectively, a 2-players coalition or the grand coalition. In order to have an efficient outcome when j is the first proposer, therefore, it is simply necessary to show that the pay-off obtainable by j calling N, say π_j^N, is higher than the one obtainable by calling a 2-players coalition, say π_j^2.
Case 1: $v(N) + v(i,k) > 2v(i,j) \wedge v(j,k) + v(i,k) > v(i,j)$. In this case, $\pi_j^N > \pi_j^2 \to v(N) - 2v(i,k) > \max(v(j,k) + 2v(i,j) - v(N) - v(i,k), 0)$. If the RHS of the inequality is equal to zero, we have seen that it is trivially satisfied since strict convexity is a sufficient condition to ensure the positiveness of π_j^N. However, even assuming $v(j,k) + 2v(i,j) - v(N) - v(i,k) > 0$, we would have $v(N) - 2v(i,k) > v(j,k) + 2v(i,j) - v(N) - v(i,k) \Rightarrow 2v(N) - 2v(i,j) - v(j,k) - v(i,k) > 0$, with the last inequality being necessarily true under strict convexity.
Case 2: $v(N) + v(i,k) \leq 2v(i,j) \wedge v(j,k) + v(i,k) \leq v(i,j)$. In this case, $\pi_j^N > \pi_j^2 \to v(N) + v(j,k) - v(i,j) - v(i,k) > v(j,k) \Rightarrow v(N) - v(i,j) - v(i,k) > 0$, again necessarily true under strict convexity.
Case 3: $v(N) + v(i,k) > 2v(i,j) \wedge v(j,k) + v(i,k) \leq v(i,j)$. In such occurrence, we have $\pi_j^N > \pi_j^2 \Rightarrow v(N) + v(j,k) - v(i,j) - v(i,k) > v(j,k) + 2v(i,j) - v(N) - v(i,k) \Rightarrow 2v(N) - 3v(i,j) > 0$. Under strict convexity we can write $v(N) = v(i,j) + v(j,k) + \eta$, with $\eta > 0$. Then, the last inequality becomes $2(v(i,j) + v(j,k) + \eta) - 3v(i,j) > 0 \Rightarrow 2v(j,k) + 2\eta - v(i,j) > 0$. If we substitute $v(i,j) + v(j,k) + \eta$ to $v(N)$ in our starting condition $v(N) + v(i,k) > 2v(i,j)$, we have $v(i,j) + v(j,k) + \eta + v(i,k) > 2v(i,j) \Rightarrow v(j,k) + v(i,k) + \eta - v(i,j) > 0$. Since $v(j,k) \geq v(i,k)$ by assumption and being η positive, then $2v(j,k) + 2\eta - v(i,j) > 0$ is necessarily true.
No more cases are possible. In fact, the case $v(N) + v(i,k) \leq 2v(i,j) \wedge v(j,k) + v(i,k) > v(i,j)$ can be excluded since, by substituting $v(N)$ with $v(i,j) + v(j,k) + \eta$ in the first inequality, we have $v(j,k) + v(i,k) + \eta \leq v(i,j) \wedge v(j,k) + v(i,k) > v(i,j)$, a contradiction. ■

Note that, while strict convexity is a sufficient condition to have an efficient outcome in a BCBM with $\alpha = 1$ whenever j is selected as first proposer, it is not a necessary one.

Proposition 4.2. *Given a strictly convex underlying coalitional game Γ following Assumption 4.1, whenever player i is selected as first proposer, the outcome will be efficient.*

Proof. The proof runs very similar to the proof of Proposition 4.1, with, *muta mutandis*, the same 3 cases being present and the forth being excluded.
Case 1: $v(N) + v(j,k) \leq 2v(i,j) \wedge v(j,k) + v(i,k) \leq v(i,j)$. In such case the inequality $\pi_i^N > \pi_i^2$, assuming $v(i,k) + 2v(i,j) - v(N) - v(j,k) > 0$, leads to the inequality $v(N) > v(i,j) + v(i,k)$, necessarily true by strict convexity. If $v(i,k) + 2v(i,j) - v(N) - v(j,k) \leq 0$, the inequality $\pi_i^N > \pi_i^2$ is necessarily true since $\pi_i^N > 0$.
Case 2: $v(N) + v(j,k) > 2v(i,j) \wedge v(j,k) + v(i,k) > v(i,j)$. In this occurrence the inequality $\pi_i^N > \pi_i^2$ reduces again to the same inequality as in Case 1: $v(N) > v(i,j) + v(i,k)$.
Case 3: $v(N) + v(j,k) > 2v(i,j) \wedge v(j,k) + v(i,k) \leq v(i,j)$. This time the inequality $\pi_i^N > \pi_i^2$ leads to $2v(N) - 3v(i,j) + v(j,k) - v(i,k) > 0$. We have already proved that $2v(N) - 3v(i,j) > 0$ in the proof of Proposition 4.1. Since $v(j,k) \geq v(i,k)$ by assumption, the inequality is necessarily satisfied. Also in this case, as mentioned, it is contradictory to have $v(N) + v(j,k) \leq 2v(i,j) \wedge v(j,k) + v(i,k) > v(i,j)$, therefore, no more cases are possible. ■

Also in the case of i being selected as first proposer in a BCBM with $\alpha = 1$, strict convexity is a sufficient condition for having efficiency, but not a necessary one. Finally, we need to consider the possibility of k being selected as first proposer.

Proposition 4.3. *Whenever player k is selected as first proposer in a BCBM whose underlying coalitional game Γ follows Assumption 4.1, strict convexity is neither a sufficient nor a necessary condition to obtain efficiency.*

Proof. By Lemma 4.1 we know that $\pi_k^2 = 0$ under both convexity and strict convexity. Therefore $\pi_k^N > \pi_k^2 \Rightarrow \pi_k^N > 0 \Rightarrow v(N) + v(j,k) > 2v(i,j) + v(i,k)$. By setting $v(N) = v(i,j) + v(j,k) + \eta$, with $\eta > 0$, we have already seen that the previous inequality translates into $\eta > v(i,j) + v(i,k) - 2v(j,k)$, whose truthfulness is independent from strict convexity. ■

Clearly, when both π_k^2 and π_k^N are lower or equal to zero, player k cannot get a strictly positive pay-off by calling any coalition, therefore she is indifferent in calling whatever coalition she belongs to: $\{k\}, \{j,k\}, \{i,k\}$ or N. If we assume that any player, when having multiple options all granting the same final pay-off, always chooses to call the coalition with the highest number of players, strict convexity becomes then a sufficient conditions for always attaining efficiency. However, this requires an additional assumption. Further note that, in presence of a non-convex underlying coalitional game with $v(i,j) + v(j,k) > v(N)$, the condition for obtaining efficiency when player k is selected as first proposer would be $2v(N) - 3v(i,j) - v(i,k) > 0$. Setting, this time, $v(N) = v(i,j) + v(j,k) - \eta$, with $\eta > 0$, we then have $-2\eta + 2v(j,k) - v(i,j) - v(i,k) > 0 \Rightarrow \eta < v(j,k) - \frac{v(i,j)+v(i,k)}{2}$.

We close this section with a table showing all the possible pay-offs configurations given a BCBM with $\alpha = 1$ and assuming an efficient outcome is reached, whoever is the selected first proposer. It is interesting to make a brief comparison with models based on discounting such as Chatterjee et al. (1993). As said before, in the limit case of $\delta = 0$, corresponding to our case of $\alpha = 1$, the grand coalition is attained simply provided that $v(N)$ is greater than the worth of any other coalition. In the present case, instead, we clearly have stricter conditions. Another important element to notice is the distribution of final pay-offs once an efficient outcome is achieved. When $\delta = 0$, all bargaining power is held by the first proposer, therefore we have a dictator-like result. The partial breakdown feature of the present model, instead, drastically changes the outcome. Only the weak player, in case she takes the responder position in the first round and $v(j,k) + v(i,k) \leq v(i,j)$ has an equilibrium pay-off equal to zero. Furthermore, as can be observed in Table 1, the"strength" of players as resulting from Assumption 4.1, is reflected in the pay-offs configuration. The advantage of being first proposer, although present, is strongly diluted, whereas the advantage of being in a stronger position according to the worth of the coalitions of belonging is fully present.

Table 1: Pay-offs for efficient outcomes in a BCBM with $\alpha = 1$

First proposer	**Equilibrium Pay-offs**	**Conditions**
k	$\pi_j = v(i,j)$ $\pi_i = v(i,j) + v(i,k) - v(j,k)$ $\pi_k = v(N) + v(j,k) - 2v(i,j) - v(i,k)$	
j	$\pi_j = v(N) + v(j,k) - v(i,j) - v(i,k)$ $\pi_i = v(i,j) + v(i,k) - v(i,j)$ $\pi_k = 0$	$v(j,k) + v(i,k) \leq v(i,j)$
j	$\pi_j = v(N) - 2v(i,k)$ $\pi_i = v(i,j) + v(i,k) - v(j,k)$ $\pi_k = v(j,k) + v(i,k) - v(i,j)$	$v(j,k) + v(i,k) > v(i,j)$
i	$\pi_j = v(i,j)$ $\pi_i = v(N) - v(i,j)$ $\pi_k = 0$	$v(j,k) + v(i,k) \leq v(i,j)$
i	$\pi_j = v(i,j)$ $\pi_i = v(N) - v(j,k) - v(i,j)$ $\pi_k = v(j,k) + v(i,k) - v(i,j)$	$v(j,k) + v(i,k) > v(i,j)$

A final remark regards the possibility to generalize our results to more than 3 players. As seen, the computation of pay-offs, even for the simple case of $\alpha = 1$, requires to consider all coalitions. Being the number of coalitions exponential in the number of players, computations become soon cumbersome. The restriction to some special classes of underlying coalitional games, such as convex games, is not of much help as seen in the analysis just shown. For the sake of clarity, therefore, we just consider the 3-players case.

5 A Generalization for 0-normalized, 3-players Games with $\alpha \in (0, 1)$

In the previous section, we have analysed a 0-normalized, 3-players BCBM for the limit case of $\alpha = 1$. The objective of the present section, instead, is to offer a general solution for all possible values of $\alpha \in (0, 1)$, with Assumption 4.1 still holding. Actually, the proposed solutions encompass the case of $\alpha = 1$, but we prefer to treat such occurrence separately due to its peculiar nature. The procedure to get these solutions will be just sketched. In fact, there is nothing complex deserving to be thoroughly explained or proved, but rather a large amount of tedious computation to be solved. For the sake of readability, we will therefore skip all intermediate passages.

In order to reach the solutions, we still apply a procedure similar to backward induction that we are now going to explain. Consider a game in which all coalitions have been burned, except for one. Assume, *w.l.o.g*, such coalition is $\{i, j\}$ and that player j is called to make a proposal. Although we already know the solution of this problem, being identical to the solution of the standard Rubinstein's alternating offers model, let us formally define the problem of j for clearness.

$$x_j = \max(v(i,j) - x_i),$$
$$\underline{x}_i = \alpha 0 + (1-\alpha)(v(i,j) - \underline{x}_j),$$
$$\underline{x}_j = \alpha 0 + (1-\alpha)(v(i,j) - \underline{x}_i),$$

where $\underline{x}_i$ and $\underline{x}_j$ are the minimum acceptable pay-offs by, respectively, player i and j. For j willing to make an acceptable proposal, we necessarily have $x_i \geq \underline{x}_i$ and, since j wants to maximize her own pay-off, $x_i = \underline{x}_i$. Therefore, we simply need to solve the simple system of two equations with two unknowns, $\underline{x}_i$ and $\underline{x}_j$, to find the equilibrium pay-offs for j and i:

$$\pi_i = x_i = \underline{x}_i = (1-\alpha)\frac{v(i,j)}{2-\alpha},$$
$$\pi_j = x_j = \frac{v(i,j)}{2-\alpha}.$$

Clearly, for $\alpha = 1$ we get $\pi_i = 0$ and $\pi_j = v(i,j)$, whereas, for $\alpha \to 0, \pi_j \approx \pi_i \approx \frac{v(i,j)}{2}$.

Once found the equilibrium for the case of a single coalition being left in the game, we consider the case of two coalitions being still active. Suppose, *w.l.o.g*, that $\{i, j\}$ and $\{j, k\}$ are the two left coalitions with $v(i,j) \geq v(j,k)$.

Notice that the ranking of players according to their strength previously proposed applies identically to the present case. Player j has now two options when taking the role of proposer, whereas i and k are still deemed to call one coalition. Let us then use as an example the case of j being called as proposer:

$$
\begin{aligned}
x_j &= \max(\max(v(i,j) - x_i), \max(v(j,k) - x_k)),\\
\underline{x}_i &= \alpha 0 + (1-\alpha)\left(v(i,j) - \underline{x}_j^{\{i,j\}}\right),\\
\underline{x}_k &= \alpha 0 + (1-\alpha)\left(v(j,k) - \underline{x}_j^{\{j,k\}}\right),\\
\underline{x}_j^{\{i,j\}} &= \alpha\left(\frac{v(j,k)}{2-\alpha}\right) + (1-\alpha)(v(i,j) - \underline{x}_i),\\
\underline{x}_j^{\{j,k\}} &= \alpha\left(\frac{v(i,j)}{2-\alpha}\right) + (1-\alpha)(v(j,k) - \underline{x}_k).
\end{aligned}
$$

Now we have two different minimally acceptable offers for j ($\underline{x}_j^{\{i,j\}}$ and $\underline{x}_j^{\{j,k\}}$), depending on which coalition she decides to call. Since she cannot mix between calling $\{i,j\}$ and $\{j,k\}$, due to our assumption of pure strategies, one of the two options is only virtual. In fact, if j decides to call coalition $\{i,j\}$ and her offer is rejected by i, in her minimal acceptable offer we still consider only the possibility for her to call again $\{i,j\}$ since, in case she rejects the offer of i and $v(i,j)$ does not get burned, the strategic situation remains unchanged compared to her previous offer. Therefore, we need to solve the problem comparing the two distinct cases of j selecting coalition $\{i,j\}$ and $\{j,k\}$ and looking at which case grants the best pay-off to j. That will be the optimal choice of j. It is easy to see that calling $\{i,j\}$ is a dominant strategy for her. The other element that is worth to mention is that j has now the possibility of getting a positive pay-off even in case she rejects an offer and the coalition gets burned. For example, if she rejects an offer of i, and coalition $\{i,j\}$ disappears, as a new proposer she is assured to get $\frac{v(j,k)}{2-\alpha}$ form what seen before. Therefore, we have $\alpha\left(\frac{v(j,k)}{2-\alpha}\right)$ in the $\underline{x}_j^{\{i,j\}}$ equation. Following is a list of equilibrium pay-offs given only coalitions $\{i,j\}$ and $\{j,k\}$ are still active and for all possible players being selected as first proposer.

j first proposer

$$
\pi_j = \frac{v(i,j)}{2-\alpha} + (1-\alpha)\frac{v(j,k)}{(2-\alpha)^2}, \qquad \pi_i = (i-\alpha)\left(\frac{v(i,j)}{2-a} - \frac{v(j,k)}{(2-\alpha)^2}\right).
$$

i first proposer

$$
\pi_i = \frac{v(i,j)}{2-\alpha} - \frac{v(j,k)}{(2-\alpha)^2}, \qquad \pi_j = (1-\alpha)\left(\frac{v(i,j)}{2-a}\right) + \frac{v(j,k)}{(2-\alpha)^2}.
$$

k first proposer

$$\pi_k = (3-2\alpha)\frac{v(j,k)}{2-\alpha} - \frac{v(i,j)}{2-\alpha}, \quad \text{if } v(j,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}, \quad \text{else } \pi_k = 0,$$
$$\pi_j = \frac{v(i,j)}{2-a} + (1-\alpha)^2\left(\frac{v(j,k)}{(2-\alpha)^2}\right).$$

Notice that, for $\alpha = 1$, it is impossible for k to obtain a positive pay-off since the condition $v(j,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ reduces to $v(j,k) > v(i,j)$, that contradicts the assumption $v(i,j) \geq v(j,k)$.

Now that we have the equilibrium solution for two coalitions left in the game, we can continue adding a third one. The procedure to find the optimal strategy for each player is basically identical to what seen till now. We need to solve a similar system of equations, once selected a first proposer, exploring each possible case at her disposal and then selecting the one granting her the highest pay-off. Whenever a proposal is rejected and the random mechanism eliminates the proposed coalition, we revert to the case of two coalitions, for which we already have the equilibrium pay-offs. Once solved for three coalitions we can add the grand coalition and explore all the remaining cases. As mentioned before, we avoid a detailed exposition of all passages that are quite cumbersome and that simply require the solution of systems of equations. Therefore, we show directly the obtained results, starting with the pay-offs obtainable by each player by calling the grand coalition in the first round. For each player, we need to distinguish four cases. The first distinction has been already encountered when discussing the results for two coalitions left in the game. We then have $v(i,k) \leq (2-\alpha)\frac{v(j,k)}{3-2\alpha}$ or $v(i,k) > (2-\alpha)\frac{v(j,k)}{3-2\alpha}$. The other two possible cases depend on the pay-off obtainable by player k in case the grand coalition gets burned. Whereas both player j and i always have a positive pay-off as proposers in absence of coalition N and given Assumption 4.1, for player k it may be impossible to get a positive pay-off. We then have the following two cases: $\pi_k^2 \leq 0$ or $\pi_k^2 > 0$, where π_k^2 indicates the pay-off obtainable by k when only 2-players coalitions are left in the game. The results are reported in Annexes A2.

In order to investigate the efficiency conditions, we need to derive the pay-offs that players can obtain by calling a 2-players coalition in the first bargaining round rather than the grand coalition. In Annexes A3 are reported the pay-offs each player can obtain by calling the most profitable 2-players coalition in the first round. Whereas in Annexes A2 we had four cases for each player, now we have six for players j and k and eight for player i. This is due to the fact that, besides the conditions $v(i,k) \leq (2-\alpha)\frac{v(j,k)}{3-2\alpha}$ or $v(i,k) > (2-\alpha)\frac{v(j,k)}{3-2\alpha}$ and $\pi_k^2 \leq 0$ or $\pi_k^2 > 0$ we have the conditions $v(i,k) \leq (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ or $v(i,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ for players j and k and $v(j,k) \leq (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ or $v(j,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ for player i. Since $v(i,j) \geq v(j,k)$, the case $v(i,k) \leq (2-\alpha)\frac{v(j,k)}{3-2\alpha} \wedge v(i,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}$ is not possible, thus reducing the number of total possibilities to six rather than eight when players j and k are considered.

In order to investigate if the outcome of a BCBM is efficient given the player selected as first proposer, it is sufficient to observe if her pay-off in Annexes A2 is greater than her pay-off provided in Annexes A3. Clearly, the comparison must be made taking into account the appropriate cases. Having already examined the efficiency conditions for $\alpha = 1$, it is interesting to look at the opposite extreme, namely $\alpha \to 0$. Following, we report the complete list of conditions assuming j is selected as first proposer. It will be further assumed that, when j calls the 2-players coalition $\{j, k\}$, in the equations defining her pay-off in Annexes A3, it always holds that $\max(..., 0) > 0$.

Efficiency conditions for $\alpha \to 1$, assuming player j is selected as first proposer.

1) $v(i,k) \leq \frac{2}{3} v(j,k) \wedge \pi_k^2 > 0$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{v(i,k)}{6},$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3} v(i,j) - \frac{v(j,k)}{2} - \frac{v(i,k)}{48}.$$
$$\text{Eff. cond.:} \quad \frac{2}{3} v(N) - \frac{v(i,j)}{3} - \frac{v(j,k)}{2} - \frac{7}{48} v(i,k) > 0.$$

2) $v(i,k) \leq \frac{2}{3} v(j,k) \wedge \pi_k^2 \leq 0$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{6} + \frac{v(j,k)}{12} - \frac{v(i,k)}{24},$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3} v(i,j).$$
$$\text{Eff. cond.:} \quad \frac{2}{3} v(N) - \frac{v(i,j)}{2} - \frac{11}{12} v(j,k) - \frac{v(i,k)}{24} > 0.$$

3) $v(i,k) > \frac{2}{3} v(j,k) \wedge \pi_k^2 > 0 \wedge v(i,k) \leq \frac{2}{3} v(i,j)$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{2}{3} v(i,k),$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3} v(i,j) - \frac{1}{3} v(j,k) + \frac{v(i,k)}{3}.$$
$$\text{Eff. cond.:} \quad \frac{2}{3} v(N) - \frac{v(i,j)}{3} - \frac{2}{3} v(j,k) - v(i,k) > 0.$$

4) $v(i,k) > \frac{2}{3} v(j,k) \wedge \pi_k^2 \leq 0 \wedge v(i,k) \leq \frac{2}{3} v(i,j)$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{6} + \frac{v(j,k)}{2} - \frac{v(i,k)}{2},$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3} v(i,j).$$
$$\text{Eff. cond.:} \quad \frac{2}{3} v(N) - \frac{v(i,j)}{2} - \frac{v(j,k)}{2} - \frac{v(i,k)}{2} > 0.$$

5) $v(i,k) > \frac{2}{3}v(i,j) \wedge \pi_k^2 > 0$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{2}{3}v(i,k),$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3}v(i,j).$$
$$\text{Eff. cond.:} \quad \frac{2}{3}v(N) - \frac{v(i,j)}{3} - v(j,k) - \frac{2}{3}v(i,k) > 0.$$

6) $v(i,k) > \frac{2}{3}v(i,j) \wedge \pi_k^2 \leq 0$

$$\pi_j^N \approx \frac{v(N)}{3} + \frac{v(i,j)}{6} + \frac{v(j,k)}{2} - \frac{v(i,k)}{2},$$
$$\pi_j^{\{j,k\}} \approx v(j,k) - \frac{v(N)}{3} + \frac{2}{3}v(i,j) - \frac{v(j,k)}{3} + \frac{v(i,k)}{3}.$$
$$\text{Eff. cond.:} \quad \frac{2}{3}v(N) - \frac{v(i,j)}{2} - \frac{v(j,k)}{6} - \frac{5}{6}v(i,k) > 0.$$

Although convexity is a sufficient condition to grant a positive pay-off to j when calling the grand coalition (π_j^N) in the first round, in each possible case, it is neither a sufficient nor a necessary condition to assure an efficient outcome. It is easy to extend such statement for the cases in which i or k are selected as first proposer.

The last thing that worth to be noticed, and that is very peculiar of this model, is that the bargaining power of players is reflected on the equilibrium pay-off they obtain when the grand coalition is attained. This is true for $\alpha = 1$, as shown in Table 1, and holds true for $\alpha \to 0$ as well. The bargaining strength obtained from belonging to worthy coalitions is retained for all possible values of α. This is a significant departure from the models based on discounting such as Chatterjee et al. (1993) and Okada (1996), where an equal split of the worth of the grand coalition is obtained for players becoming extremely patient, and provided such allocation is inside the Core of the game. The shift from discounting to partial breakdown, therefore, entails a less egalitarian outcome even for asymptotic values of α. If we take as an example the case in which $v(i,k) \leq \frac{2}{3}v(j,k)$ and $\pi_k^2 > 0$, we can clearly see as the rank of players' pay-offs reflects their rank in bargaining power. This is true both when we compare the pay-offs obtained as first proposer, and when the comparison addresses the pay-offs as responders.

Pay-offs for $\alpha \to 0$ assuming $v(i,k) \leq \frac{2}{3}v(j,k) \wedge \pi_k^2 > 0$ and the attainment of an efficient outcome

j first proposer

$$\pi_j \approx \tfrac{v(N)}{3} + \tfrac{v(i,j)}{3} - \tfrac{v(i,k)}{6},$$
$$\pi_i \approx \tfrac{v(N)}{3} + \tfrac{v(i,j)}{3} - \tfrac{v(j,k)}{2} + \tfrac{v(i,k)}{12}, \quad \pi_k \approx \tfrac{v(N)}{3} - \tfrac{2}{3}v(i,j) + \tfrac{v(j,k)}{2} + \tfrac{v(i,k)}{12}.$$

i **first proposer**

$$\pi_i \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{v(j,k)}{2} + \frac{v(i,k)}{12},$$

$$\pi_j \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{v(i,k)}{6}, \quad \pi_k \approx \frac{v(N)}{3} - \frac{2}{3}v(i,j) + \frac{v(j,k)}{2} + \frac{v(i,k)}{12}.$$

k **first proposer**

$$\pi_k \approx \frac{v(N)}{3} - \frac{2}{3}v(i,j) + \frac{v(j,k)}{2},$$

$$\pi_j \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{v(i,k)}{12}, \quad \pi_i \approx \frac{v(N)}{3} + \frac{v(i,j)}{3} - \frac{v(j,k)}{2} + \frac{v(i,k)}{12}.$$

6 Conclusions

This present paper has presented a variant of the partial breakdown bargaining model for coalitional games: the Burning coalition Bargaining Model (BCBM). Such model can be described as a Rubinstein-type model where discounting, or cost of bargaining, is substituted by the possibility of breakdown of negotiations. Its peculiarity mainly resides in the type of breakdown being implemented. Differently from two- or multi-players bargaining, where breakdown implies the end of negotiations, leaving players with their outside options pay-offs, and differently from the partial breakdown à la Hart and Mas-Colell (1996), that implies the exclusion of some players, here it implies the dissolution of the worth of the coalition that has been selected. Since in Hart and Mas-Colell (1996), the exclusion of a player implies the loss of all coalitions she belongs to, we can say that the present paper displays an even more partial type of breakdown, being only one coalition per time to be "burned".

Apart from this peculiar feature, the model adopts a standard rejecter-proposes assumption and equal recognition probabilities. The analysis focuses on 0-normalized, 3-players coalitional games, with a special attention placed on strictly convex games. The simple adoption of a different version of partial breakdown is sufficient to produce results that are markedly distant from models based on discounting and even from other types of partial breakdown. When the probability of breakdown is equal to one, corresponding to a value of zero for discounting, achieving efficiency is more problematic in the present model than in models with discounting. The advantage of being first proposer is diluted by the presence of more options for each player, thus generally granting a more equal pay-offs distribution. When an efficient outcome is achieved, still the final distribution of pay-offs reflects the bargaining strength of players, where strength here means to belong to more worthy coalitions. This peculiar feature is retained even for values of the probability of breakdown approaching the opposite extreme, as to say zero. Although the Shapley value, supported by the partial breakdown models à la Hart and Mas-Colell (1996), also rewards players belonging to worthy coalitions, here it is more important the relation between the worth of coalitions a player belongs to and the one she is not part of.

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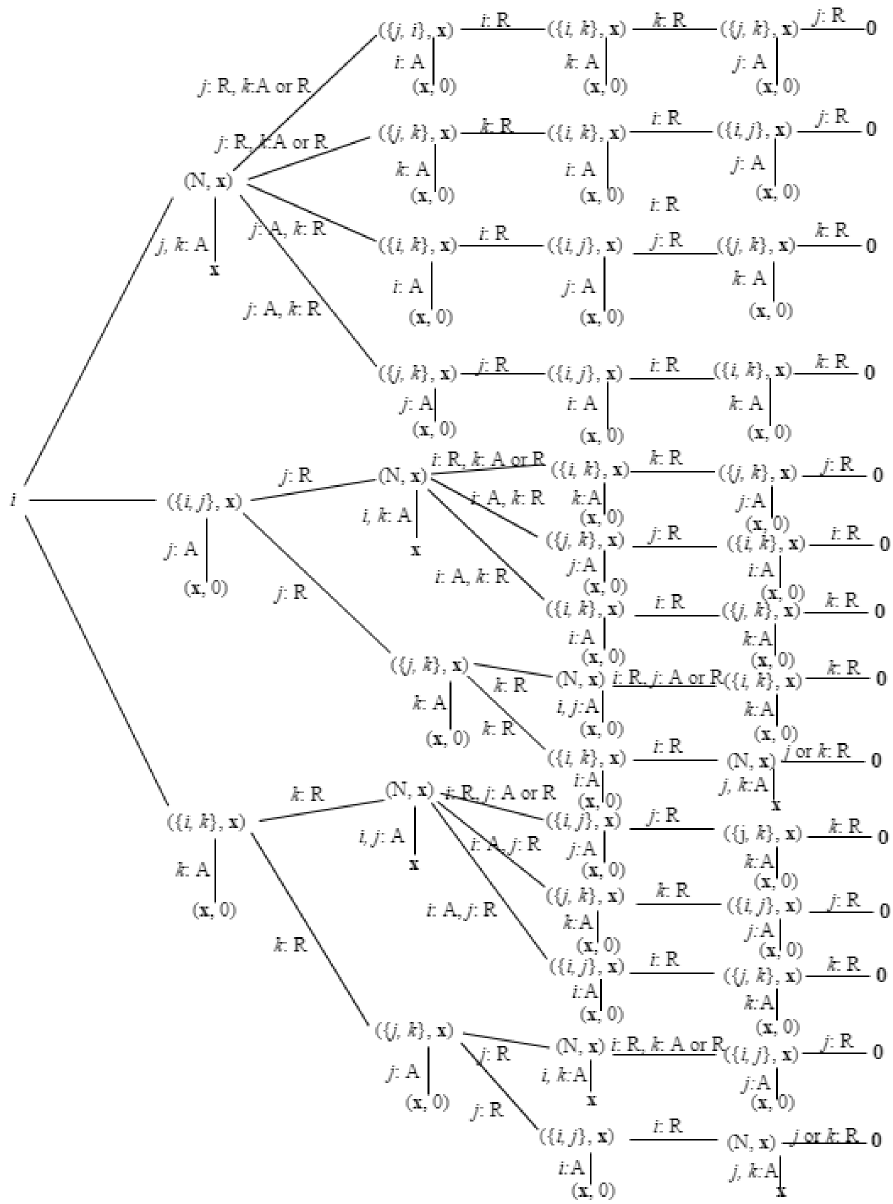
Annexes

A1

Figure 1: Extensive form representation of a 3-players BCBM with $\alpha = 1$.
i selected as first proposer



A2

j first proposer

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_j = &\quad \frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{2\alpha(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)^2} - \frac{2(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_i = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{3-2\alpha} - \frac{(6-8\alpha+3\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2} + \frac{(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_k = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{(4-3\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{(6-6\alpha+\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2} + \frac{(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0$$

$$\begin{aligned}
\pi_j = &\quad \frac{v(N)}{3-2\alpha} + \frac{(2-6\alpha+3\alpha^2)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{1}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right),\\
\pi_i = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(5-9\alpha+7\alpha^2-2\alpha^3)}{(3-2\alpha)}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right),\\
\pi_k = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{(1-\alpha)(4-3\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + (1-\alpha)\frac{(4-5\alpha+2\alpha^2)}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right).
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_j = &\quad \frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{2(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_i = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)} - \frac{(4-3\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} + \frac{(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_k = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{v(i,j)}{(3-2\alpha)} + \frac{v(j,k)}{(3-2\alpha)} + \frac{(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0$$

$$\begin{aligned}
\pi_j = &\quad \frac{v(N)}{3-2\alpha} + \frac{(2-6\alpha+3\alpha^2)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{v(j,k)}{(2-\alpha)} - \frac{2(12-25\alpha+16\alpha^2-8\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^4},\\
\pi_i = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{v(j,k)}{(2-\alpha)} + \frac{(12-21\alpha+12\alpha^2-2\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_k = &\quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{(1-\alpha)(4-3\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{2\alpha(1-\alpha)(2-3\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4}.
\end{aligned}$$

i first proposer

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_i = &\ \frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(12-16\alpha+5\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^3} + \frac{(4-8\alpha+3\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4},\\
\pi_j = &\ \frac{(1-\alpha)(v(N)+v(i,j))}{3-2\alpha} + \frac{(1-\alpha)}{3-2\alpha}\left(\frac{\alpha v(j,k)}{(2-\alpha)^2} + \frac{(4-\alpha)v(i,k)}{(2-\alpha)^3}\right),\\
\pi_k = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{(4-3\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{(12-18\alpha+8\alpha^2-\alpha^3)v(j,k)}{(3-2\alpha)(2-\alpha)^3} + \frac{(4-6\alpha+4\alpha^2-\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^4}.
\end{aligned}$$

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0$$

$$\begin{aligned}
\pi_i = &\ \frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(1-\alpha)(7-3\alpha)}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^3} - \frac{v(i,k)}{(2-\alpha)^4}\right),\\
\pi_j = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(1-\alpha)(4-\alpha)}{(3-2\alpha)}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right),\\
\pi_k = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(1-\alpha)(1-3\alpha+\alpha^2)}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^3} - \frac{v(i,k)}{(2-\alpha)^4}\right).
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_i = &\ \frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{2v(j,k)}{3-2\alpha} + \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_j = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)+v(j,k)}{3-2\alpha} - \frac{(16-28\alpha+17\alpha^2-4\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_k = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{(4-3\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{v(j,k)}{3-2\alpha} - \frac{(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0$$

$$\begin{aligned}
\pi_i = &\ \frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{v(j,k)}{2-\alpha} + \frac{(10-15\alpha+8\alpha^2-2\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_j = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{v(j,k)}{2-\alpha} - \frac{(12-19\alpha+10\alpha^2-2\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_k = &\ \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{2(1-\alpha)^2v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

k first proposer

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_k =& \quad \frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(12-23\alpha+15\alpha^2-3\alpha^3)v(j,k)}{(3-2\alpha)(2-\alpha)^3} + \frac{\alpha(1-3\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4},\\
\pi_j =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{3-2\alpha} + \frac{\alpha(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha^3)} - \frac{(1-\alpha)(4-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^4},\\
\pi_i =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{v(i,j)}{3-2\alpha} - \frac{(6-8\alpha+3\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2} + \frac{(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0$$

$$\begin{aligned}
\pi_k =& \quad \frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{(3-2\alpha)(2-\alpha)} + \alpha\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right),\\
\pi_j =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{(1-\alpha)(3-\alpha)}{(3-2\alpha)}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right),\\
\pi_i =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} - \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(3-3\alpha+\alpha^2)}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right).
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}
\pi_k =& \quad \frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_j =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)+v(j,k)}{3-2\alpha} - \frac{(16-28\alpha+17\alpha^2-4\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_i =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{3-2\alpha} - \frac{(4-3\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} + \frac{(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0$$

$$\begin{aligned}
\pi_k =& \quad \frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2\alpha(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_j =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{v(j,k)}{2-\alpha} - \frac{(12-19\alpha+10\alpha^2-2\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3},\\
\pi_i =& \quad \frac{(1-\alpha)v(N)}{3-2\alpha} + \frac{v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{v(j,k)}{2-\alpha} + \frac{(12-21\alpha+12\alpha^2-2\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^3}.
\end{aligned}$$

A3

j **first proposer** (calling $\{j,k\}$)

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(12-23\alpha+15\alpha^2-3\alpha^3)v(j,k)}{(3-2\alpha)(2-\alpha)^3} + \frac{(1-3\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4}, 0\right).\end{aligned}$$

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \alpha\Big(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\Big), 0\right).\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(i,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0 \wedge v(i,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2\alpha(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)}, 0\right).\end{aligned}$$

$$\pi_k^2 > 0 \wedge v(i,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-9\alpha+4\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).\end{aligned}$$

$$\pi_k^2 \leq 0 \wedge v(i,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_j = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-9\alpha+4\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2\alpha(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)}, 0\right).\end{aligned}$$

i **first proposer** (calling $\{i,k\}$)

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(j,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(12-23\alpha+15\alpha^2-3\alpha^3)v(j,k)}{(3-2\alpha)(2-\alpha)^3} + \frac{(1-3\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4}, 0\right).\end{aligned}$$

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0 \wedge v(j,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \alpha\Big(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\Big), 0\right).\end{aligned}$$

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(j,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-8\alpha+4\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(12-23\alpha+15\alpha^2-3\alpha^3)v(j,k)}{(3-2\alpha)(2-\alpha)^3} - \frac{\alpha(1-3\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^4}, 0\right).\end{aligned}$$

$$v(i,k) \leq (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0 \wedge v(j,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-8\alpha+4\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \alpha\Big(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\Big), 0\right).\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(j,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)^2} - \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \leq 0 \wedge v(j,k) \leq (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_i = \quad & v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{\alpha v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2\alpha(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).\end{aligned}$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(j,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\pi_i = v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-8\alpha+4\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)^2} - \frac{4(1-\alpha)(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0 \wedge v(j,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\pi_i = v(i,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{(6-8\alpha+4\alpha^2)v(j,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} - \frac{2v(i,j)}{3-2\alpha} + \frac{2\alpha(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).$$

k **first proposer** (calling $\{j,k\}$)

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0$$

$$\pi_k = v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{3(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} + \frac{2\alpha(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)^2} - \frac{2(2-2\alpha+\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right).$$

$$v(i,k) \le (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0$$

$$\pi_k = v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{3(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{(2-6\alpha+3\alpha^2)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{1}{3-2\alpha}\left(\frac{v(j,k)}{(2-\alpha)^2} - \frac{v(i,k)}{(2-\alpha)^3}\right), 0\right).$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 > 0 \wedge v(i,k) \le (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\pi_k = v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{3(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{2(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).$$

$$v(i,k) > (2-\alpha)\tfrac{v(j,k)}{3-2\alpha} \wedge \pi_k^2 \le 0 \wedge v(i,k) \le (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\pi_k = v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{3(1-\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{(2-6\alpha+3\alpha^2)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{v(j,k)}{2-\alpha} - \frac{2(12-25\alpha+16\alpha^2-8\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^4}, 0\right).$$

$$\pi_k^2 > 0 \wedge v(i,k) > (2-\alpha)\tfrac{v(i,j)}{3-2\alpha}$$

$$\pi_k = v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(6-5\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{2(1-\alpha)v(j,k)}{(3-2\alpha)(2-\alpha)} - \frac{2(8-12\alpha+5\alpha^2)v(i,k)}{(3-2\alpha)(2-\alpha)^3}, 0\right).$$

$$\pi_k^2 \leq 0 \wedge v(i,k) > (2-\alpha)\frac{v(i,j)}{3-2\alpha}$$

$$\begin{aligned}\pi_k = \quad & v(j,k) - \alpha \max\left(\frac{v(N)}{3-2\alpha} + \frac{(1-2\alpha)v(i,j)}{(3-2\alpha)(2-\alpha)} - \frac{(6-5\alpha)v(i,k)}{(3-2\alpha)(2-\alpha)^2}, 0\right) + \\ & -(1-\alpha)\max\left(\frac{v(N)}{3-2\alpha} + \frac{(2-6\alpha+3\alpha^2)v(i,j)}{(3-2\alpha)(2-\alpha)^2} + \frac{v(j,k)}{2-\alpha} - \frac{2(12-25\alpha+16\alpha^2-8\alpha^3)v(i,k)}{(3-2\alpha)(2-\alpha)^4}, 0\right).\end{aligned}$$