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Pay package reshuffling and managerial incentives: A principal-agent analysis

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Abstract

By deferring a significant portion of managers' remuneration, managers bear the risk of their choices for a longer period of time and avoid excessive risk taking. The effectiveness of this mechanism is jeopardized if managers reshuffle their pay packages; this is possible when trades in the components of pay packages are not verifiable. In this paper, we investigate the relevance of trade verifiability in pay packages design. We analyze a principal-agent model with agent's compensation made of different commodities which can be exchanged on competitive markets at given prices. We consider both the case when trades in commodities are verifiable, and when they are not. We prove that an optimal contract when trades are verifiable remains optimal when trades are not verifiable if agent's preferences for commodities are independent of the action performed. We provide examples to illustrate what happens when preferences' independence fails.

JEL Classification: D82 (Asymmetric and Private Information • Mechanism Design), D86 (Economics of Contract: Theory), J33 (Compensation Packages • Payment Methods).

Keywords: Pay package reshuffling, Principal-agent model, Independent preferences.

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1. Introduction

Practitioners and scholars are lively debating the role of compensation policies in promoting excessive risk taking and "short-termism" (i.e., increasing firms' current stock value at the expense of long-run value), by executives of both financial and non financial companies (see, e.g., Donnan and Fleming 2015, Fleming 2014, and International Monetary Fund 2014). Some propose caps on pay to disicpline managers (see, e.g., Barker 2013 and Arnold and Agnew 2015; see also Chou and Chen 2015, who study the effects of pay caps in a principal-agent model). Many criticize the structure of compensations based on equity-like instruments that guarantee managers a large share of profits in case of success of risky projects, while leaving other stakeholders to bear losses in case of failure (see, e.g., Edmans 2010). In particular, as argued by Skapinker (2015), "one of the early ways in which companies attempted to overcome [agency problems] was to award executives share options. [...] The problem was that too many executives sold their shares as soon as they exercised the options. Critics argued that this encouraged excessive risk-taking as executives attempted to boost the share price when the options came due."

Experts have suggested different mechanisms to reduce the incentives to take excessive risks. Since the negative consequences of managers' decisions may take several years to materialize, these mechanisms share the common principle that managers should bear the risk of their choices for a longer period of time. Therefore, a significant share of managers' compensation should be deferred and made contingent on the firm's future financial conditions, so that it can be withheld if the firm goes bankrupt (see, for example, French et al. 2010, chapter 6). However, as pointed out by the same group of experts, "holdbacks only reduce management's incentives to take excessive risks if management cannot hedge its deferred compensation. Any hedging of deferred compensation should therefore be prohibited." (see Squam Lake Group 2013). To enforce this requirement, managers' trades in the components of pay packages must be verifiable. If this condition is not met, the effectiveness of deferred compensation is jeopardized.

Given the relevance of trade (un)verifiability for policy prescriptions, in this paper, we investigate its impact on managerial incentives. To this end, we frame our analysis as a principal-agent model with the standard assumption that the agent's action (a manager's choice of a risky vs. a safe project, for example) cannot be verified by the principal (other stakeholders, for example).[1] Furthermore, we introduce two non-standard features. Firstly, the agent's compensation consists in multiple commodities; with this assumption, we capture the idea that managers' pay package may contain different "commodities" like, for example deferred and equity-like income. Secondly, the agent can trade these commodities; with this assumption, we capture the idea that managers can reshuffle pay packages, for example by hedging deferred compensation.

To see the effects of trade (un)verifiability, consider the benchmark case where the principal is able to verify the agent's trades in commodities. In this case, the terms of the contract can be set so that the agent is bound consuming a specific bundle. As a consequence, the only choice left to agent is to perform the action which is optimal given that bundle, and the principal can induce her preferred action by a suitable commodity bundle. Suppose instead that the principal cannot verify the agent's trades in commodities. In this case, the agent can choose a different bundle and a different action. If this happens, the principal may not be able to induce her preferred action by proposing the same compensation as when trades are verifiable.

To illustrate this point, consider the following back-of-the-envelope example, which echoes the debate on executive compensation: a manager can choose a risky project or a safe project. The manager's pay package may contain equity-like compensation and deferred compensation. If the manager chooses the safe project, he gets utility from deferred compensation only; if the manager chooses the risky project, he gets utility from equity-like compensation only. Therefore, when the pay package contains more deferred compensation, the manager prefers the safe project; when the pay package contains more equity-like compensation, the manager prefers the risky project.

In Figure 1 (left panel), points in the shaded area represent pay packages with more deferred compensation than equity-like compensation; points in the white area represent pay packages

[1] In what follows, we refer to the principal as "she" and to the agent as "he".

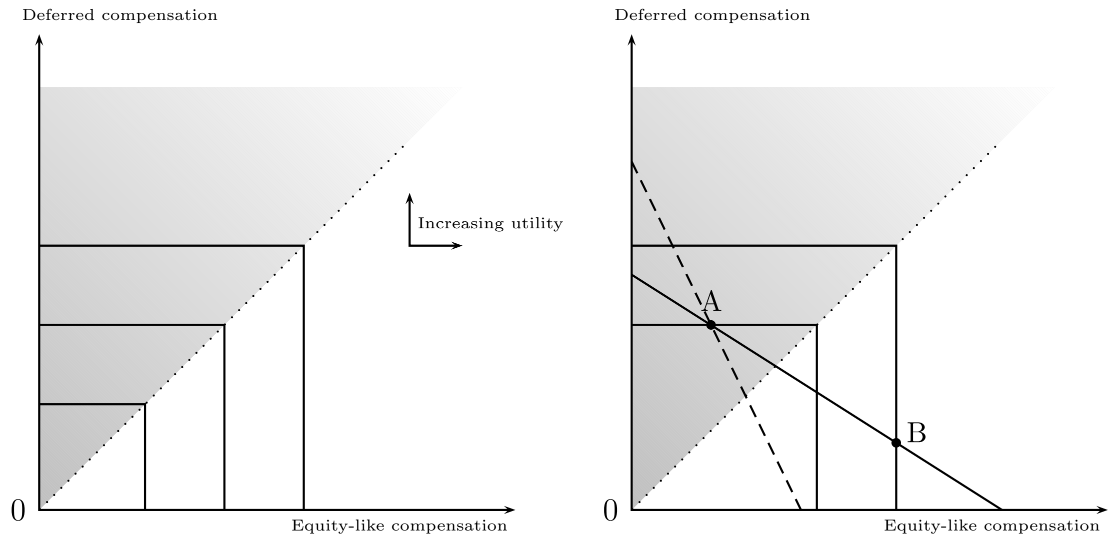

Figure 1: Back-of-the-envelope example.

with more equity-like compensation than deferred compensation; the inverted L-shaped curves represent pay packages that provide the manager with the same utility when the choice of the project is taken into account.[2]

In this example, it seems easy to incentivize the manager to select the safe project when this is the other stakeholders' preferred choice: simply grant him more deferred compensation, for example as in the package corresponding to point A in Figure 1 (right panel).

Yet, suppose that it is not possible to prevent the manager from trading deferred compensation for equity-like income, due to lack of verifiability of trades. In this case, two scenarios can arise. Assume first that the term of trades are favorable to equity-like income, so that the manager can reshuffle his pay package along the solid downward sloping line in Figure 1 (right panel). Under this scenario, the manager can substitute deferred compensation with equity-like income, and choose the risky project. This corresponds to a move from point A to point B: since the manager obtains much equity-like income in exchange for deferred compensation, this double deviation increases the manager's utility, and it indeed occurs. Assume now that the terms of trades are not favorable to equity-like income, so that the manager can reshuffle his

[2]Packages in the shaded (white) area induce the manager to select the safe (risky) project, hence their utility corresponds to the amount of deferred (equity-like) compensation as measured by the horizontal (vertical) branches of the inverted L-shaped curves.

pay package along the dashed downward sloping line in Figure 1 (right panel). In this case, no points in the white area corresponding to utility higher than that obtained by the manager at A could be reached, hence no double deviation will occur.

In the first scenario, preventing the manager from hedging his deferred compensation is crucial to induce the choice of the safe project, hence trade (un)verifiability is an issue. In the second scenario, such provision is not relevant, hence (un)verifiability is not an issue. A crucial feature of this example is that the manager's preferences over the two types of compensation depend on the project choice.

In this paper, we show that, if preferences over pay packages components are independent of the manager's choice, it is irrelevant to prevent managers from reshuffling pay packages. The intuition for this result is summarized as follows: when trades are verifiable, the principal proposes a contract that specifies a bundle and prescribes an action, which is optimal given that bundle. If trades are not verifiable, the agent can trade the prescribed bundle for a different one of equal value, and change action. However, under the independence assumption, the agent will never be willing to do so for any terms of trade.

Since the independence assumption does not hold in the agency relationships considered in the executive compensations' debate, our result provides a general argument for the relevance of trade (un)verifiability.

Overall, the contribution of our paper is twofold. On the theoretical ground, we generalize traditional analyses of the principal-agent relationships which assume from the outset that there exists a single commodity, referred to as money or income. In this case, the possibility of trades, whether verifiable or not, is excluded by construction. Moreover, the common assumption of additively-separable preferences implies that the choice of action is independent of preferences for money. Therefore, our result reveals that a single-commodity agency model with separability can be interpreted as a reduced form of a model with multiple commodities whose choice is not verifiable.

On the policy ground, we provide two related arguments for the relevance of trade (un)verifiability in the executive compensation debate. The first (direct) argument is illustrated in the above example, as well as in further examples discussed later on. The second (indirect) argument is given by our result based on the independence hypothesis and proved in a general setting. We believe that our analysis can offer a convincing justification for scholars and practitioners to put the (un)verifiability issue at the center of the debate on executive compensation.

The paper is organized as follows: in the next section we discuss the related literature; the model is described in section 3, while the main results regarding the irrelevance of verifiability of trades are presented in section 4. Our concluding comments are presented in section 5. Furthermore, appendix A contains the proofs of the auxiliary results. Finally, appendix B contains an extension of the model to the case of uncertainty.

2. Related literature

The issue of (un)verifiable actions and trades has been explored in models both of partial equilibrium and of general equilibrium. Among the former, some papers address the issue of how to characterize the optimal contract between a principal and an agent when two different forms of compensation are available: a (numeraire) good interpreted as money and another good interpreted as perks or fringe benefit which cannot be traded on competitive markets (see e.g. Zou 1996, Marino and Zaboinik 2008, and Bennardo et al. 2010). By specifying the degree of complementarity or substitutability between effort and the perks, the optimal contract is characterized by means of first order conditions. A more recent strand of partial equilibrium models focuses on issues related to managerial compensation (see Edmans and Gabaix, 2009, for a survey). Among them, Edmans and Liu (2011) investigate when inside debt is superior solution to agency problems than equity-like instruments. Edmans et al. (2012) develop a dynamic model of executive compensation and characterize the optimal contract, made of corporate stocks and cash payments, that induces proper incentives in managers' effort, private saving, and short-termism. Hoffmann, Inderst and Opp (2015) propose a principal agent setting to investigate the impact of deferred incentive pay on the diligence with which agents conduct their business. They derive conditions under which deferred compensation leads to lower diligence.

In models of general competitive equilibrium with moral hazard, it is known that allowing consumers to engage in anonymous trades may interfere with constrained efficiency of competitive equilibrium (see e.g. Greenwald and Stiglitz 1986, Arnott, Grennwald and Stiglitz 1994).[3] However, it is also known that, if trades are permitted, constrained efficiency generally occurs provided that preferences for commodities are independent of the action performed (see e.g. Lisboa 2001, Panaccione 2007, Kielnthong and Townsend 2011, Acemoglu and Simsek 2012).[4,5] Indeed, in these models the link between (in)dependence and (in)efficiency is related to the possibility of simultaneous change in consumption and action by the agent akin to the one described in the introductory example.

3. The model

In our model, a principal is interested in hiring an agent to perform some action beneficial to her. The principal proposes to the agent a contract which specifies a compensation and a recommended action. If the contract is refused, the agent has an outside option, whose utility value is taken as given by both parties.

We assume that the action can be observed, but not verified. Therefore, the compensation cannot be made contingent on this variable. We also suppose that the agent's trades in commodities are observable; however, as mentioned in the introduction, we consider two cases as regards to the possibility to verify them.

In one case, we assume that the principal can verify agent's trades in commodities, so that she can impose a given commodity bundle by enforceable covenants; in the introductory example, this scenario corresponds to banning any hedging of managers' deferred compensation. If the contract is accepted, the agent consumes the prescribed bundle and chooses the action which is optimal for himself.

[3] For a similar result, see also Citanna and Villanacci (2002).

[4] This property is also relevant for the undesirability of differential commodity taxation in models with moral hazard, see e.g. Arnott and Stiglitz (1983/1986) and Panaccione and Ruscitti (2010).

[5] The issues raised by (un)verifiable trades have also been investigated in the literature on dynamic optimal taxation with privately observed shocks (see e.g. Kocherlakota 2005, Albanesi and Sleet 2006, Golosov and Tsyvinski 2006).

In the polar opposite, we assume that the agent has unverifiable access to markets for commodities; in the introductory example, this scenario corresponds to the possibility for the manager to reshuffle his pay package. In this case, we suppose that the contract specifies a compensation. If the contract is accepted, the agent chooses a bundle whose value, at market prices, is not greater than that of the compensation, and perform the action which is optimal for himself.

In both cases, given the proposed contract, the agent's choices are correctly anticipated by the principal.

3.1. Notation and assumptions

Let $X = \mathbb{R}^L_+$ be the consumption set, and $x \in X$ a generic bundle. We assume a finite number $L \geqslant 2$ of commodities. Let $\mathcal{A}$ and a be the set of actions and a generic action, respectively. We suppose that $\mathcal{A}$ is a compact subset of $\mathbb{R}_+$. The agent's utility from a bundle x and an action a is $u(x, a)$. We assume that u is a continuous real-valued function, and that it is locally non-satiated on X for every $a \in A$. Let p denote the vector of prices of commodities, which are taken as given by the principal and by the agent; we assume that $p \in \mathbb{R}^L_{++}$.

In what follows, the above assumptions will be referred to as the *standard assumptions*. In addition, we posit the following assumption:

Assumption I (**Independence**): *agent's preferences for $x \in X$ are independent of $a \in \mathcal{A}$, hence u can be written as $u(x, a) = g(f(x), a)$.*

By assumption I every bundle $x \in X$ can be assigned a sub-utility level $f(x)$ independent of a. Therefore, if x is preferred to x' for a given action a, so that $f(x) \geqslant f(x')$, then it will be preferred given any other action $a' \neq a$. When u is differentiable, assumption I implies that the marginal rate of substitution between any two commodities is independent of a.[6]

Finally, let $\varrho(a)$ denote the gross monetary benefit for the principal when the agent chooses the action a. Given an agent's monetary compensation w, the net monetary benefit for the

[6] Seminal references on independence are Debreu (1959) and Blackorby et al. (1978, 2008).

principal is $\varrho(a) - w$.

3.2. The Problems of the Agent

In this section, we introduce two problems that characterize the agent's behavior and capture the different contractual clauses that the principal's can impose depending on whether commodity trades are verifiable or not.

Suppose that trades are not verifiable, so that the agent can choose an action a and a bundle x whose monetary value is not greater than $w > 0$. In this case, he will solve the following problem:

$$\max_{x \in X,\, a \in \mathcal{A}} \; u(x,a) \quad \text{s.t.} \quad p \cdot x \leqslant w\,, \tag{A1}$$

whose non-empty set of solutions is $\varphi(p,w) \subset X \times \mathcal{A}$. Suppose instead that trades are verifiable, so that the agent can only choose an action a, since he is bound to consume a bundle $x \in X$ prescribed by the contract. In this case, he will solve the following problem:

$$\max_{a \in \mathcal{A}} \; u\left(x,a\right)\,, \tag{A2}$$

whose non-empty set of solutions is $a\left(x\right) \subset \mathcal{A}$.

3.3. The Problems of the Principal

In this section, we introduce two problems that characterize the optimal contract proposed by the principal when trades are verifiable and when trades are not verifiable.

Let u denote the utility level of the agent's best available outside option. Moreover, let $u_\circ := u(x_\circ, a)$ for $a \in a(x_\circ)$, with $x_\circ = 0$, denote the level of utility the agent can secure himself when level of consumption is the minimum admissible. The contract proposed by the principal is accepted only if it provides the agent with a level of utility at least equal to u; in what follows, we suppose $u > u_\circ$, which is included in our standard assumptions.

Assume first that commodity trades are verifiable. In this case, the contract prescribes a bundle x and recommends an action $a \in a(x)$. If accepted, the contract results in the gross monetary benefit $\varrho(a)$ and in the cost $p\cdot x$, which is the market value of the bundle. To maximize her net benefit, the principal solves the following problem:

$$\max_{x,a} \quad \varrho(a) - p \cdot x \tag{P1}$$

$$x \in X\,, \tag{1a}$$

$$a \in a(x)\,, \tag{1b}$$

$$u(x,a) \geqslant u \tag{1c}$$

whose set of solutions is $\beta(p,u) \subset X \times \mathcal{A}$.

Suppose instead that commodity trades are not verifiable. In this case, if the contract prescribes a bundle x, the agent cannot be prevented from trading it for a different one whose value is at most $p \cdot x$, and simultaneously change action. Therefore, in this scenario it is without loss of generality to assume that the contract specifies a monetary compensation and recommends some $(x,a) \in \varphi(p,w)$. To maximize her net benefit, the principal solves the following problem:

$$\max_{x,a,w} \quad \varrho(a) - w \tag{P2}$$

$$w \geqslant 0\,, \tag{2a}$$

$$(x,a) \in \varphi(p,w)\,, \tag{2b}$$

$$u(x,a) \geqslant u \tag{2c}$$

whose set of solutions is $\gamma(p,u) \subset X \times A \times \mathbb{R}$.

4. On the irrelevance of verifiability of trades

In this section, we prove that, when assumption I holds, a solution (x^*,a^*) to principal's problem (P1) is also a solution to principal's problem (P2) with $w = p\cdot x^*$. Therefore, under preferences independence, the contract that maximizes the principal's net benefit when trades are verifiable also maximizes principal's net benefit when trades are *not* verifiable.

Hence, when assumption I holds, the principal can always choose her optimal contract *as if* trades were verifiable. However, since this assumption is violated in the agency relationships considered in the executive compensation debate, this results provides a general (indirect) argument for the relevance of trade (un)verifiability.

4.1. Preliminary results

Before we embark in the proof of the main results, we prove some preliminary results invoked later on.[7]

Lemma 1. *Suppose assumption I holds. Then* $g(f(\hat{x}), a) > g(f(x), a)$ *implies* $f(\hat{x}) > f(x)$ *and* $g(f(\hat{x}), a) = g(f(x), a)$ *implies* $f(\hat{x}) = f(x)$ *for any* $x, \hat{x} \in X$ *and* $a \in \mathcal{A}$.

Lemma 1 follows since g is an increasing function of f as implied by assumption I. Now pick $w > 0$ and consider the following conditional utility maximization problem and conditional expenditure minimization problem:

$$\max_{x \in X} u(x, a) \quad \text{s.t.} \quad p \cdot x \leqslant w \tag{A3}$$

$$\min_{x \in X} p \cdot x \quad \text{s.t.} \quad u(x, a) \geqslant u \tag{A4}$$

Let $x(p, w, a)$ and $h(p, u, a)$ denote the non-empty set of solutions to, respectively, problem (A3) and (A4). The next lemma states that if (x, a) is a solution to problem (A1), then a is a solution to problem (A2) given x, and x is a solution to problem (A3) given a.

Lemma 2. *Suppose the standard assumptions hold. Then* $(x, a) \in \varphi(p, w)$ *implies* $a \in a(x)$ *and* $x \in x(p, p \cdot x, a)$.

The next two lemmas record two straightforward implications of assumption I. Lemma 3 states that, if two bundles give the same level of sub-utility f, then they induce the same sets of

[7]The (standard) proofs of these results are provided, for sake of completeness, in appendix A.

optimal actions. Similarly, lemma 4 states that if a bundle is optimal conditional on an action $a \in \mathcal{A}$, then it is optimal given any other action $\hat{a} \neq a$.

Lemma 3. *If the standard assumptions and assumption I hold, then $a(x) = a(\hat{x})$ for all $x, \hat{x} \in X$ such that $f(x) = f(\hat{x})$*

Lemma 4. *If the standard assumptions and assumption I hold, then $x \in x(p, w, a)$ implies $x \in x(p, w, \hat{a})$ for any $a, \hat{a} \in \mathcal{A}$*

The following lemmas propose two useful results concerning the conditional expenditure minimization problem (A4) and the utility maximization problem (A1).

Lemma 5. *If the standard assumptions hold, $x \in h(p, u, a)$ implies $u(x, a) = u$.*

Lemma 6. *If the standard assumptions hold and $w > 0$, $(x, a) \in \varphi(p, w)$ implies $p \cdot x = w$.*

4.2. Main results

The next two propositions state the central results of our analysis. In particular, proposition 1 shows that, if a contract (x^*, a^*), which is optimal for the principal when trades are verifiable, is a solution to the agent's maximization problem when he can choose both the bundle x and the action a, then the same contract, with monetary compensation $w^* = p \cdot x^*$, is optimal for the principal when trades are not verifiable.

Proposition 1. *Suppose the standard assumptions hold. If $(x^*, a^*) \in \beta(p, u)$ solves (A1) with $w = p \cdot x^*$, then $(x^*, a^*, p \cdot x^*) \in \gamma(p, u)$.*

Proof. Pick $(x^*, a^*) \in \beta(p, u)$ such that it is also a solution to (A1), hence $(x^*, a^*) \in \varphi(p, p \cdot x^*)$ by assumption. In this case, if $(x^*, a^*, p \cdot x^*) \notin \gamma(p, u)$ then there exists $(\hat{x}, \hat{a}, \hat{w})$, which satisfies (2a) $-$ (2c), and such that $\varrho(\hat{a}) - \hat{w} > \varrho(a^*) - p \cdot x^*$. However, since $(\hat{x}, \hat{a}) \in \varphi(p, \hat{w})$, $\hat{x} \in X$ and, by lemma 2, $\hat{a} \in a(\hat{x})$. Therefore, $(\hat{x}, \hat{a})$ satisfies (1a) $-$ (1c), hence $\varrho(a^*) - p \cdot x^* \geqslant \varrho(\hat{a}) - p \cdot \hat{x} = \varrho(\hat{a}) - \hat{w}$, where the equality holds because of lemma 6. This contradiction implies $(x^*, a^*, p \cdot x^*) \in \gamma(p, u)$. $\square$

Proposition 1 isolates the condition that guarantees that an optimal contract when trades are verifiable is the principal's best choice even when trades are not verifiable. The next proposition shows that, when assumption I holds, the general condition identified in proposition 1, i.e. $(x^*, a^*) \in \beta(p, u)$ solves (A1), is always true.

Proposition 2. *Suppose the standard assumptions hold. If assumption I holds, then any $(x^*, a^*) \in \beta(p, u)$ solves (A1) with $w = p \cdot x^*$.*

Proof. We prove the proposition in three steps.

Step 1: If $(x^*, a^*) \in \beta(p, u)$, that is if (x^*, a^*) is a solution to (P1), then x^* solves (A4) with $a = a^*$ and $u = u(x^*, a^*)$. Suppose there exists $(x, a) \in \beta(p, u)$ such that $x \notin h(p, u, a)$, with $u = u(x, a)$. In this case, there is some $\tilde{x} \in X$ such that $u(\tilde{x}, a) \geqslant u = u(x, a)$ and $p \cdot \tilde{x} < p \cdot x$. Pick $\hat{x} \in h(p, u, a)$, so that $p \cdot \hat{x} \leqslant p \cdot \tilde{x} < p \cdot x$ and $u(\hat{x}, a) = u = u(x, a)$, where the last equality follows from lemma 5. In this case, $f(\hat{x}) = f(x)$ by virtue of lemma 1, hence $a(\hat{x}) = a(x)$ by virtue of lemma 3. Since $a \in a(x)$, the latter equality implies $a \in a(\hat{x})$. It follows that $(\hat{x}, a)$ satisfies (1a)$-$(1c) and therefore $\varrho(a) - p \cdot x \geqslant \varrho(a) - p \cdot \hat{x}$. However, $p \cdot \hat{x} < p \cdot x$ implies $\varrho(a) - p \cdot \hat{x} > \varrho(a) - p \cdot x$, which is a contradiction.

Step 2: If x^* solves (A4) with $a = a^*$ and $u = u(x^*, a^*)$, then x^* solves (A3) with $a = a^*$ and $w = p \cdot x^*$. Suppose there exists $x \in h(p, u, a)$, with $u = u(x, a)$ such that $x \notin x(p, p \cdot x, a)$. In this case, there is some $\tilde{x} \in X$ such that $p \cdot \tilde{x} \leqslant p \cdot x$ and $u(\tilde{x}, a) > u(x, a) = u$. Pick $\hat{x} \in x(p, p \cdot x, a)$, so that $p \cdot \hat{x} \leqslant p \cdot x$ and $u(\hat{x}, a) \geqslant u(\tilde{x}, a) > u(x, a) = u$. By continuity of the utility function and of the inner product it follows that $p \cdot \delta\hat{x} < p \cdot x$ and $u(\delta\hat{x}, a) \geqslant u$ for δ close enough to one. However, $x \in h(p, u, a)$ and $u(\delta\hat{x}, a) \geqslant u$ implies $p \cdot x \leqslant p \cdot \delta x$. This is a contradiction, and therefore $x \in h(p, u, a)$ implies $x \in x(p, p \cdot x, a)$.[8]

Step 3: If x^* solves (A3) with $a = a^*$ and $w = p \cdot x^*$, and if a^* solves (A2) with $x = x^*$, then (x^*, a^*) solves (A1) with $w = p \cdot x^*$. Suppose there exists (x, a) such that $x \in x(p, p \cdot x, a)$, $a \in a(x)$, and $(x, a) \notin \varphi(p, p \cdot x)$. Since (x, a) is feasible in (A1), $(x, a) \notin \varphi(p, p \cdot x)$ implies that there exists $\tilde{x} \in X$ and $\tilde{a} \in \mathcal{A}$ such that $p \cdot \tilde{x} \leqslant p \cdot x$ and $u(\tilde{x}, \tilde{a}) > u(x, a)$. Pick $(\hat{x}, \hat{a}) \in \varphi(p, p \cdot x)$,

[8] Notice that this step does not require assumption I.

so that $u(\hat{x},\hat{a}) \geqslant u(\tilde{x},\tilde{a}) > u(x,a)$, and, by lemma 2, $\hat{a} \in a(\hat{x})$ and $\hat{x} \in x(p, p{\cdot}x, \hat{a})$. Furthermore, $\hat{x} \in x(p, p \cdot x, a)$ by lemma 4. Since $x \in x(p, p \cdot x, a)$, this implies $u(\hat{x}, a) = u(x, a)$. Therefore, $f(\hat{x}) = f(x)$ by lemma 1, and hence $a(\hat{x}) = a(x)$ by virtue of lemma 3. Since $a \in a(x)$, this implies $a \in a(\hat{x})$, whence $u(\hat{x},\hat{a}) = u(\hat{x},a)$ and, therefore, $u(\hat{x},\hat{a}) = u(x,a)$. However, this contradicts $u(\hat{x},\hat{a}) > u(x,a)$. $\square$

Proposition 1 and proposition 2 are important as they identify the condition that guarantees that the lack of verifiable information on commodity trades has no impact on the principal's choice. When assumption I holds, the principal chooses the optimal contract as if trades were observable.

By contrast, when assumption I is not satisfied, as in the introductory back-of-the-envelope example, it may be possible that an optimal contract for the principal when trades are verifiable is not the agent's optimal choice when he can reshuffle the bundle and change action. Therefore, such a contract would not be a feasible choice for the principal when trades are not verifiable, i.e. it would not satisfy constraint (2b) in problem (P2), and, a fortiori, it cannot be the principal's best choice in this scenario. The next two examples provide further illustrations of this case; in both examples, the left panel corresponds to a failure of the third step in the proof of proposition 2.

Example 1. *Let* $L = 2$ *and* $\mathcal{A} = \{a_1, a_2\}$. *Moreover, let* $u(x, a_1) = \min\{x_1, \delta x_2\}$ *and* $u(x, a_2) = \min\{\delta x_1, x_2\}$ *for some* $\delta > 1$. *In this case, if* x *is such that* $x_1 < x_2$, *then* $u(x, a_1) = x_1 < \min\{\delta x_1, x_2\} = u(x, a_2)$, *hence* $a(x) = \{a_2\}$. *By the same token, if* $x_2 < x_1$, *then* $a(x) = \{a_1\}$. *Finally, if* x *is such that* $x_2 = x_1$, *then* $a(x) = \{a_1, a_2\}$. *Therefore, for any bundle* x *in the shaded (white) area in Figure 2 below,* a_2 (a_1) *is optimal for the agent. In the figure, the downward sloping budget line is given by* $x_2 = w/p_2 - x_1 p_1/p_2$, *while the piece-wise linear curves are the locus of commodity bundles such that* $u(x, a(x))$ *is constant. Consider the left panel and suppose that* (x, a_2) *is the solution to problem (P1). As it is apparent from the picture,* (x, a_2) *is not a solution to problem (A2) with* $w = p \cdot x$. *Indeed the latter is* $(\hat{x}, a_1)$. *Consider now the right panel and suppose that* $(\tilde{x}, a_1)$ *is the solution to problem (P1). In this case, it is apparent*

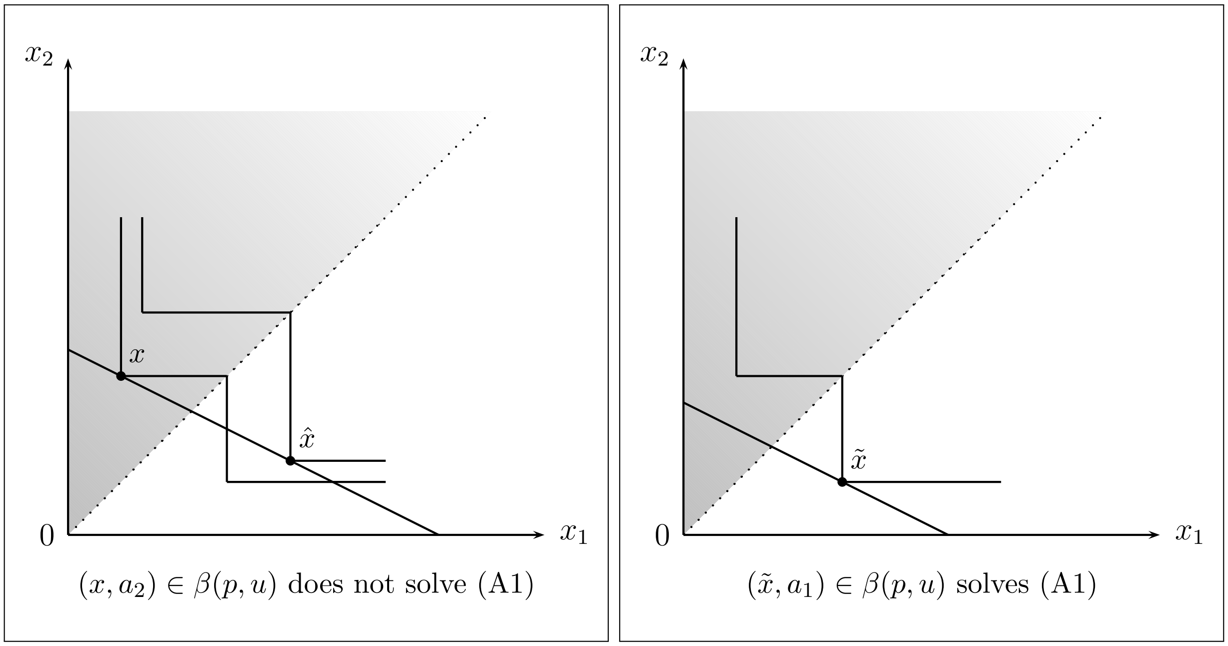

Figure 2: Illustration of example 1.

that it is also the solution to problem (A2) with $w = p \cdot \tilde{x}$.

Example 2. *Let* $L = 2$ *and* $\mathcal{A} = [0, 1]$. *Moreover, let* $u(x, a) = \left[a x_1^\delta + (1 - a) x_2^\delta\right]^{1/\delta}$ *for* $\delta \neq 0$ *and* $u(x, a) = x_1^a x_2^{1-a}$ *for* $\delta = 0$. *For any given value of* δ, *if* x *is such that* $x_2 > x_1$, *then* $a(x) = \{0\}$, *while if* $x_2 < x_1$, *then* $a(x) = \{1\}$.[9] *Finally, if* x *is such that* $x_2 = x_1$, *then* $a(x) = [0, 1]$. *Therefore, for any bundle* x *in the shaded (white) area in Figure 3 below,* $a = 0$ *(*$a = 1$*) is optimal for the agent. In the figure, the downward sloping budget line is given by* $x_2 = w/p_2 - x_1 p_1/p_2$, *while the piece-wise linear curves are the locus of commodity bundles such that* $u(x, a(x))$ *is constant. Consider the left panel and suppose that* $(x, 0)$ *is the solution to problem (P1). As it is apparent from the picture,* $(x, 0)$ *is not a solution to problem (A2) with* $w = p \cdot x$. *Indeed the latter is* $(\hat{x}, 1)$. *Consider now the right panel and suppose that* (x, a_1) *is the solution to problem (P1). In this case, it is apparent that it is also the solution to problem (A2) with* $w = p \cdot x$.

[9]If $x_1 \neq x_2$, from a standard property of the power mean it follows that $u(x, a) < \max\{x_1, x_2\}$ whenever $a \in (0, 1)$.

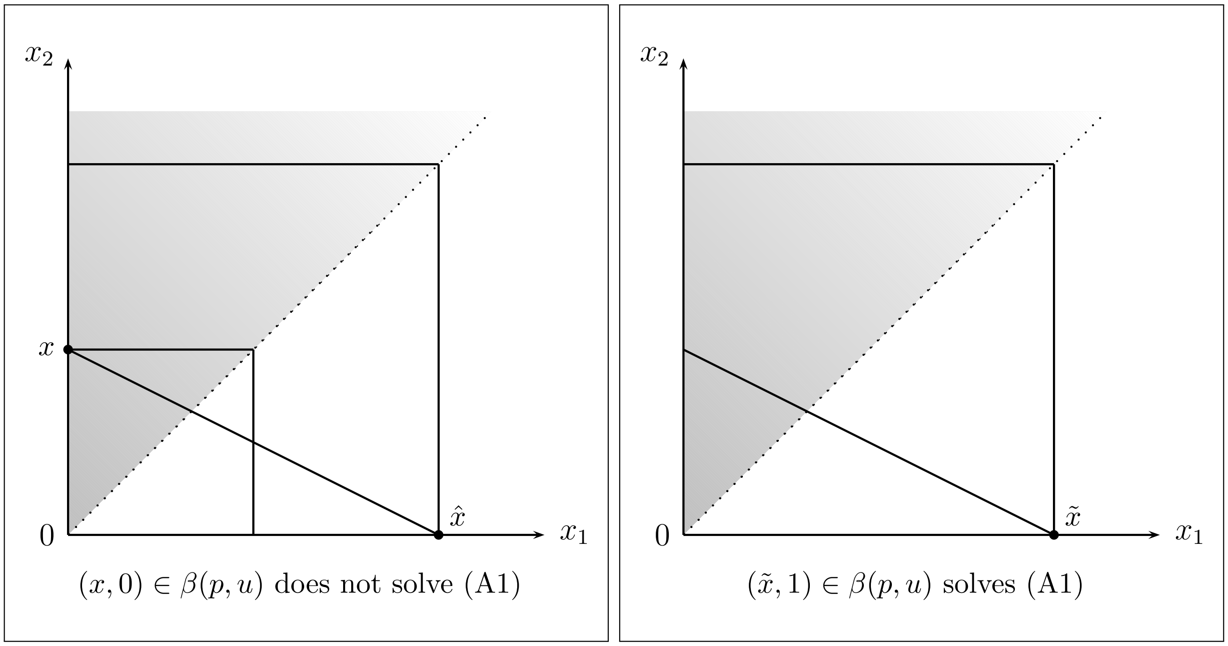

Figure 3: Illustration of example 2.

5. Conclusions

We have analyzed a principal-agent model in which the agent derives direct utility from a bundle of commodities, which can be exchanged on competitive markets at given prices. We have considered both the case in which the principal can verify these trades, and the case when she cannot. After having argued that, in the latter case, the alternatives available to the principal can be significantly restricted, we have investigated when a contract, which is optimal when trades are verifiable, would be optimal even if they were not verifiable. Our main result is to show that this happens when agent's preferences for commodities are independent of the action performed.

From a theoretical perspective, this result generalizes most of the existing analyzes of the principal-agent relationship which assume from the outset that there exists only one commodity, and provides a possible foundation for the common assumption of additively-separable preferences between money and action.

From a policy perspective, our analysis aims to contribute to the lively debate on managers' compensations by putting the trade (un)verifiability issue at the center of the stage.

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Appendix A

Proof of lemma 1. Suppose there exist $\hat{x}, x \in X$ such that $g(f(\hat{x}), a) > g(f(x), a)$ and $f(\hat{x}) \leqslant f(x)$. If $f(\hat{x}) = f(x)$, then $g(f(\hat{x}), a) = g(f(x), a)$, which contradicts $g(f(\hat{x}), a) > g(f(x), a)$. Similarly, if $f(\hat{x}) < f(x)$, then $g(f(\hat{x}), a) < g(f(x), a)$, since assumption I implies that g is increasing in f (see, for instance, Luenberger 1995, p.117). Again by virtue of assumption I, $f(\hat{x}) \neq f(x)$ implies $g(f(\hat{x}), a) \neq g(f(x), a)$, and this is equivalent to the second implication stated in the lemma. □

Proof of lemma 2. To prove the first implication, suppose there exists $(x, a) \in \varphi(p, w)$ such that $a \notin a(x)$. In this case, there exists $\hat{a} \in \mathcal{A}$ such that $u(x, \hat{a}) > u(x, a)$. However, $(x, a) \in \varphi(p, w)$ implies $u(x, a) \geqslant u(x, \hat{a})$, which is a contradiction. To prove the second implication, suppose there exists $(x, a) \in \varphi(p, w)$ such that $x \notin x(p, w, a)$. In this case, there exists $\hat{x} \in X$ such that $p \cdot \hat{x} \leqslant w$ and $u(\hat{x}, a) > u(x, a)$. However, $(x, a) \in \varphi(p, w)$ implies $u(x, a) \geqslant u(\hat{x}, a)$, which is a contradiction. □

Proof of lemma 3 Since the proof is straightforward, we leave it to the reader.

Proof of lemma 4. Suppose there exists $a, \hat{a} \in \mathcal{A}$ such that $x \in x(p, w, a)$ and $x \notin x(p, w, \hat{a})$. In this case, there exists $\hat{x} \in X$ such that $p \cdot \hat{x} \leqslant w$ and $u(\hat{x}, \hat{a}) > u(x, \hat{a})$. By lemma 1, $f(\hat{x}) > f(x)$, and therefore $u(\hat{x}, a) > u(x, a)$ by assumption I. However, $x \in x(p, w, a)$ implies $u(x, a) \geqslant u(\hat{x}, a)$, which is a contradiction. □

Proof of lemma 5. Suppose there exists $x \in h(p, u, a)$ such that $u(x, a) > u$. Since $u > u_\circ$, $x \neq 0$, hence $p \cdot x > 0$. Therefore, $\delta x \in X$ whenever $\delta \in (0, 1)$. By virtue of the continuity of the utility function and of the inner product, $u(\delta x, a) \geqslant u$ and $p \cdot \delta x < p \cdot x$ whenever δ is close enough to one. However, $x \in h(p, u, a)$ and $u(\delta x, a) \geqslant u$ imply $p \cdot x \leqslant p \cdot \delta x$, which is a contradiction. □

Proof of lemma 6. Suppose there exists $(x, a) \in \varphi(p, w)$ such that $p \cdot x < w$. In this case, local non-satiation and continuity of the inner product imply that there exists $\hat{x} \in X$ such that $p \cdot \hat{<} w$ and $u(\hat{x}, a) > u(x, a)$. However, since $(x, a) \in \varphi(p, w)$, it follows that $u(x, a) \geqslant u(x, a)$, which is a contradiction. □

Appendix B

In this appendix, we provide a generalization of our analysis to the case of uncertainty, which is important since it makes our model more comparable with standard principal-agent models with hidden action. Interestingly, we find that the results of propositions 1 and 2 are robust to the introduction of uncertainty.

We assume a finite number of states of the world, $s = 1, 2, ...S$, and a finite number $L \geqslant 2$ of commodities in every state. In every s, the consumption set is $X = \mathbb{R}^L_+$, and a generic state-contingent bundle is $x_s \in X$. As before, $\mathcal{A}$ and a denote the set of actions and a generic action, respectively. We retain the assumption that $\mathcal{A}$ is a compact subset of $\mathbb{R}_+$. In every state, the agent's utility is $u(x_s, a)$. We suppose that $u : X \times \mathcal{A} \to \mathbb{R}$ is a function continuous and, for every $a \in A$, locally non-satiated on X.

The probability of each state is affected by the action performed by the agent. Let $\pi_s(a)$ denotes the probability of state s when action a is chosen. We assume that $\pi_s : \mathcal{A} \to \mathbb{R}_{++}$ is a continuous function for every s. Clearly, $\sum_s \pi_s(a) = 1$ for every a. Let $\chi = (x_s, x_{-s}) \in X^S$, in which $x_{-s} = (x_1, ...x_{s-1}, x_{s+1}, ..., x_S) \in X^{S-1}$. We invoke the standard expected utility hypothesis and we let $v(\chi, a) = \sum_s \pi_s(a)u(x_s, a)$ denote the utility of a bundle (χ, a). Observe that $v : X^S \times \mathcal{A} \to \mathbb{R}$ is a continuous functions.

We assume that commodity prices are state *in*dependent and strictly positive, so that $p \in \mathbb{R}^L_{++}$ in every state s; furthermore, we let $p \otimes \chi = (p \cdot x_1, ..., p \cdot x_S) \in \mathbb{R}^S$ for $\chi \in X^S$. These are the *standard assumptions* in this section.

Furthermore, the assumption of independence of preferences is consistently adapted as follows:

Assumption I' (Independence): *In every state $s = 1, ..., S$, agent's preferences for $x_s \in X$ are independent of $a \in \mathcal{A}$, hence u can be written as $u(x_s, a) = g(f(x_s), a)$.*

Finally, let $\varrho_s(a)$ denote the state-s gross monetary benefit for the principal when the agent chooses the action a, and $\sum_s \pi_s(a)\varrho_s(a)$ denote the expected net benefit.

We extend the agent's and the principal's problems to accommodate for the uncertainty. Therefore, problem (A1) and (A2) become, respectively,

$$\max_{\chi, a} \quad v(\chi, a) \tag{A1'}$$

$$x_s \in X \quad \text{for } s = 1, ..., S\,, \tag{3a}$$

$$p \cdot x_s \leqslant \omega_s \quad \text{for } s = 1, ..., S\,, \tag{3b}$$

$$a \in \mathcal{A}\,, \tag{3c}$$

with non-empty set of solutions $\varphi(p, \omega) \subset X^S \times \mathcal{A}$, in which $\omega = (\omega_1, ..., \omega_S)$, and

$$\max_{a \in \mathcal{A}} v(\chi, a)\ , \tag{A2'}$$

with non-empty set of solutions $a(\chi) \subset \mathcal{A}$. Similarly, problem (A3) and (A4) become, respectively,

$$\max_{x_s \in X} v(x_s, x_{-s}, a) \quad \text{s.t.} \quad p \cdot x_s \leqslant \omega_s \tag{A3'}$$

$$\min_{x_s \in X} \; p \cdot x_s \quad \text{s.t.} \quad v(x_s, x_{-s}, a) \geqslant v \tag{A4'}$$

in which $\omega_s > 0$ and $(x_{-s}, a) \in X^{S-1} \times \mathcal{A}$ are given.

Given a compensation for the agent equal to ω_s in state s, the expected net benefit for the principal is $\sum_s \pi_s(a)(\varrho_s(a) - \omega_s)$. As it is standard, we assume that states are verifiable, so that compensation can be state-contingent. However, we retain the assumption that actions are not verifiable. Therefore, problem (P1) and problem (P2) become, respectively,

$$\max_{\chi, a} \quad \sum_s \pi_s(a) \left(\varrho_s(a) - p \cdot x_s\right) \tag{P1'}$$

$$x_s \in X \quad \text{for } s = 1, ..., S \,, \tag{4a}$$

$$a \in a\left(\chi\right) \,, \tag{4b}$$

$$v\left(\chi, a\right) \geqslant v \,, \tag{4c}$$

with set of solutions $\beta(p, v) \subset X^S \times \mathcal{A}$, and

$$\max_{\chi, a, \omega} \quad \sum_s \pi_s(a) \left(\varrho_s(a) - \omega_s\right) \tag{P2'}$$

$$\omega_s \geqslant 0 \quad \text{for } s = 1, ..., S \,, \tag{5a}$$

$$(\chi, a) \in \varphi(p, \omega) \,, \tag{5b}$$

$$v\left(\chi, a\right) \geqslant v \,, \tag{5c}$$

with set of solutions $\gamma(p, v) \subset X^S \times A \times \mathbb{R}^S$. Similarly to the case of certainty, let $v_\circ := v(\chi_\circ, a)$ for $a \in a(\chi_\circ)$ and $\chi_\circ = 0$, and include $v > v_\circ$ in the standard assumption for this section.

For sake of exposition, in this appendix both lemmas and propositions are proved after their statements. The following results generalize lemma 1 to lemma 6

Lemma 7. *Suppose assumption I' holds. Then $g(f(\hat{x}_s), a) > g(f(x_s), a)$ implies $f(\hat{x}_s) > f(x_s)$ and $g(f(\hat{x}_s), a) = g(f(x_s), a)$ implies $f(\hat{x}_s) = f(x_s)$.*

Proof: Suppose there exist $\hat{x}_s, x_s \in X$ such that $g(f(\hat{x}_s), a) > g(f(x_s), a)$ and $f(\hat{x}_s) \leqslant f(x_s)$. If $f(\hat{x}_s) = f(x_s)$, then $g(f(\hat{x}_s), a) = g(f(x_s), a)$, which contradicts $g(f(\hat{x}_s), a) > g(f(x_s), a)$. Similarly, if $f(\hat{x}_s) < f(x_s)$, then $g(f(\hat{x}_s), a) < g(f(x_s), a)$, since assumption I' implies that g is increasing in f. Again by virtue of assumption I', $f(\hat{x}_s) \neq f(x_s)$ implies $g(f(\hat{x}_s), a) \neq g(f(x_s), a)$, and this is equivalent to the second implication stated in the lemma. □

Lemma 8. *Suppose the standard assumptions hold. Then $(\chi, a) \in \varphi(p, \omega)$ implies $a \in a(\chi)$ and $x_s \in x_s(p, \omega_s, x_{-s}, a)$ for every s.*

Proof: To prove the first implication, suppose there exists $(\chi, a) \in \varphi(p, \omega)$ such that $a \notin a(\chi)$. In this case, there exists $\hat{a} \in \mathcal{A}$ such that $v(\chi, \hat{a}) > v(\chi, a)$. However, $(\chi, a) \in \varphi(p, \omega)$ implies

$v(\chi, a) \geqslant v(\chi, \hat{a})$, which is a contradiction. To prove the second implication, suppose there exists $(\chi, a) \in \varphi(p, \omega)$ such that $x_s \notin x_s(p, \omega_s, x_{-s}, a)$ for some s. In this case, there exists $\hat{x}_s \in X$ such that $p \cdot \hat{x}_s \leqslant \omega_s$ and $v(\hat{x}_s, x_{-s}, a) > v(\chi, a)$. However, $(\chi, a) \in \varphi(p, w)$ implies $v(\chi, a) = v(x_s, x_{-s}, a) \geqslant v(\hat{x}_s, x_{-s}, a)$, which is a contradiction. □

Lemma 9. *Suppose the standard assumptions and assumption I' hold. Then* $a(\chi) = a(\hat{\chi})$ *for all* $\chi, \hat{\chi} \in X^S$ *such that* $f(x_s) = f(\hat{x}_s)$ *for every* s.

Proof: Since the proof is straightforward, we leave it to the reader.

Lemma 10. *Suppose the standard assumptions and assumption I' hold. Then* $x_s \in x_s(p, \omega_s, x_{-s}, a)$ *implies* $x_s \in x_s(p, \omega_s, x_{-s}, \hat{a})$ *for any* $a, \hat{a} \in \mathcal{A}$.

Proof: Suppose there exist $a, \hat{a} \in \mathcal{A}$ such that $x_s \in x_s(p, \omega_s, x_{-s}, a)$ and $x_s \notin x_s(p, \omega_s, x_{-s}, \hat{a})$. In this case, there exists some $\hat{x}_s \in X$ such that $p \cdot \hat{x}_s \leqslant \omega_s$ and $v(\hat{x}_s, x_{-s}, \hat{a}) > v(x_s, x_{-s}, \hat{a})$. The former inequality implies $v(x_s, x_{-s}, a) \geqslant v(\hat{x}_s, x_{-s}, a)$, since $x_s \in x_s(p, \omega_s, x_{-s}, a)$, and hence $u(x_s, a) \geqslant u(\hat{x}_s, a)$ by virtue of the expected utility hypothesis. However, $v(\hat{x}_s, x_{-s}, \hat{a}) > v(x_s, x_{-s}, \hat{a})$ implies $u(\hat{x}_s, \hat{a}) > u(x_s, \hat{a})$ and therefore $f(\hat{x}_s) > f(x_s)$ by virtue of lemma 7. From assumption I', it follows that $u(\hat{x}_s, a) > u(x_s, a)$, which is a contradiction. □

Lemma 11. *Suppose the standard assumptions hold. Then* $x_s \in h_s(p, v, x_{-s}, a)$ *implies* $v(x_s, x_{-s}, a) = v$.

Proof: Suppose there exists $x_s \in h_s(p, v, x_s, a)$ such that $v(x_s, x_{-s}, a) > v$. Since $v > v_\circ$, $x \neq 0$ and hence $p \cdot x_s > 0$. Therefore, $\alpha x \in X$ whenever $\alpha \in (0, 1)$. By virtue of the continuity of the utility function and of the inner product, $v(\alpha x_s, x_s, a) \geqslant v$ and $p \cdot \alpha x_s < p \cdot x_s$ when α is close enough to one. However, $x_s \in h_s(p, v, x_{-s}, a)$ and $v(\alpha x_s, x_{-s}, a) \geqslant v$ imply $p \cdot x_s \leqslant p \cdot \alpha x_s$, which is a contradiction. □

Lemma 12. *Suppose the standard assumptions hold, and let* $\omega_s > 0$ *for every* s. *Then* $(\chi, a) \in \varphi(p, \omega)$ *implies* $p \cdot x_s = \omega_s$ *for every* s.

Proof: Suppose there exists $(\chi, a) \in \varphi(p, \omega)$ such that $p \cdot x_s < \omega_s$ for some s. In this case, local non-satiation and continuity of the inner product imply that there exists $\hat{x}_s \in X$ such that $p \cdot \hat{x}_s < \omega_s$ and $u(\hat{x}_s, a) > u(x_a, s)$. Since $\pi_s(a) > 0$, it follows that $v(\hat{x}_s, x_{-s}, a) > v(x_s, x_{-s}, a) = v(\chi, a)$, which is a contradiction. □

The following lemma describes an implication of the expected utility hypothesis that will be used to generalize our main results.

Lemma 13. *Suppose the standard assumptions hold. Then* $x_s \in x_s(p, \omega_s, x_{-s}, a)$ *implies* $x_s \in x_s(p, \omega_s, \hat{x}_{-s}, a)$ *for any* $x_{-s}, \hat{x}_{-s} \in X^{S-1}$

Proof: Suppose $x_{-s}, \hat{x}_{-s} \in X^{S-1}$ are such that $x_s \in x_s(p, \omega_s, x_{-s}, a)$ and $x_s \notin x_s(p, \omega_s, \hat{x}_{-s}, a)$. In this case, there exists some $\hat{x}_s \in X$ such that $p \cdot \hat{x}_s \leqslant \omega_s$ and $v(\hat{x}_s, \hat{x}_{-s}, a) > v(x_s, \hat{x}_{-s}, a)$. The former inequality implies $v(x_s, x_{-s}, a) \geqslant v(\hat{x}_s, x_{-s}, a)$, since $x_s \in x_s(p, \omega_s, x_{-s}, a)$, and hence $u(x_s, a) \geqslant u(\hat{x}_s, a)$ by virtue of the expected utility hypothesis. However, $v(\hat{x}_s, \hat{x}_{-s}, a) > v(x_s, \hat{x}_{-s}, a)$ also implies $u(\hat{x}_s, a) > u(x_s, a)$, and this is a contradiction. □

The main results are generalized as follows:

Proposition 3. *Suppose the standard assumptions hold. If $(\chi^*, a^*) \in \beta(p, v)$ solves (A1') with $\omega_s = p \cdot x_s^*$ for every s, then $(\chi^*, a^*, p \otimes \chi^*) \in \gamma(p, u)$.*

Proof. Pick $(\chi^*, a^*) \in \beta(p, v)$. Since $(\chi^*, a^*) \in \varphi\,(p, p \otimes \chi^*)$ by assumption, $(\chi^*, a^*, p \otimes \chi^*) \notin \gamma(p, v)$ implies that there exist $(\hat{\chi}, \hat{a}, \hat{\omega})$, which satisfies (5a) $-$ (5c), and $\sum_s \pi_s(\hat{a})\,(\varrho_s(\hat{a}) - \hat{\omega}_s) > \sum_s \pi_s(a^*)\,(\varrho_s(a^*) - p \cdot x_s^*)$. Since $(\hat{\chi}, \hat{a}) \in \varphi(p, \hat{\omega})$, $\hat{x}_s \in X$ for every s and, by lemma 8, $\hat{a} \in a(\hat{\chi})$. It follows that $(\hat{\chi}, \hat{a})$ satisfies (4a) $-$ (4c), and therefore $\sum_s \pi_s(a^*)\,(\varrho_s(a^*) - p \cdot x_s^*) > \sum_s \pi_s(\hat{a})\,(\varrho_s(\hat{a}) - p \cdot \hat{x}_s)$. However, $\sum_s \pi_s(\hat{a})\,(\varrho_s(\hat{a}) - p \cdot \hat{x}_s) = \sum_s \pi_s(\hat{a})\,(\varrho_s(\hat{a}) - \hat{\omega}_s)$ by lemma 12. Hence, $\sum_s \pi_s(a^*)\,(\varrho_s(a*) - p \cdot x_s^*) > \sum_s \pi_s(\hat{a})\,(\varrho_s(\hat{a}) - \hat{\omega}_s)$, which is a contradiction. Therefore, we conclude that $(\chi^*, a^*, p \otimes \chi^*) \in \gamma(p, u)$. □

Proposition 4. *Suppose the standard assumptions hold. If assumption I' holds, then $(\chi^*, a^*) \in \beta(p, v)$ solves (A1') with $\omega_s = p \cdot x_s^*$ for every s.*

Proof. We prove the result in three steps.

Step 1: If $(\chi^*, a^*) \in \beta(p, v)$, then, for every s, x_s^* solves (A4') with $a = a^*$, $x_{-s} = x_{-s}^*$ and $v = v(\chi^*, a^*)$. Suppose there exists $(\chi, a) \in \beta(p, v)$ such that $x_s \notin h_s(p, v, x_{-s}, a)$ for some s, with $v = v(\chi, a) = v(x_s, x_{-s}, a)$. In this case, there is some $\tilde{x}_s \in X$ such that $v(\tilde{x}_s, x_{-s}, a) \geqslant v = v(x_s, x_{-s}, a)$ and $p \cdot \tilde{x}_s < p \cdot x_s$. Pick $\hat{x}_s \in h_s(p, v, x_{-s}, a)$, so that $p \cdot \hat{x}_s \leqslant p \cdot \tilde{x}_s < p \cdot x_s$ and $v(\hat{x}_s, x_{-s}, a) = v = v(x_s, x_{-s}, a)$ by lemma 11. Because of the expected utility assumption, $v(\hat{x}_s, x_{-s}, a) = v(x_s, x_{-s}, a)$ implies $u(\hat{x}_s, a) = u(x_s, a)$. Therefore, $f(\hat{x}_s) = f(x_s)$ by lemma 7, hence $a(\hat{x}_s, x_{-s}) = a(x_s, x_{-s})$ by virtue of lemma 9. Since $a \in a(x_s, x_{-s})$, the latter equality implies $a \in a(\hat{x}_s, x_{-s})$. It follows that $(\hat{x}_s, x_{-s}, a)$ satisfies (4a)$-$(4c) and therefore $\sum_s \pi_s(a)(\varrho_s(a) - p \cdot x_s) \geqslant \pi_s(a)(\varrho_s(a) - p \cdot \hat{x}_s) + \sum_{s' \neq s} \pi_s(a)(\varrho_{s'}(a) - p \cdot x_{s'})$. This is equivalent to $\varrho_s(a) - p \cdot x_s \geqslant \varrho_s(a) - p \cdot \hat{x}_s$, hence to $p \cdot x_s \leqslant p \cdot \hat{x}_s$, which is a contradiction.

Step 2: For any $(\chi^*, a^*) \in X^S \times \mathcal{A}$ and for every s, if x_s^* solves (A4') with $a = a^*$, $x_{-s} = x_{-s}^*$ and $v = v(x_s^*, x_{-s}^*, a^*)$, then x_s^* solves (A3'), with $a = a^*$, $x_{-s} = x_{-s}^*$ and $\omega_s = p \cdot x_s^*$. Suppose there exists, for some s, $x_s \in h_s(p, v, x_{-s}, a)$ such that $x_s \notin x_s(p, p \cdot x_s, x_{-s}, a)$. In this case, there is some $\tilde{x}_s \in X$ such that $p \cdot \tilde{x}_s \leqslant p \cdot x_s$ and $v(\tilde{x}, x_{-s}, a) > v(x_s, x_{-s}, a)$. Pick $\hat{x}_s \in x_s(p, p \cdot x_s, x_{-s}, a)$, so that $p \cdot \hat{x}_s \leqslant p \cdot x_s$ and $v(\hat{x}_s, x_{-s}, a) \geqslant v(\tilde{x}_s, x_{-s}, a) > v(x_s, x_{-s}, a)$. By lemma 11, it follows that $p \cdot \alpha \hat{x}_s < p \cdot x_s \leqslant p \cdot \alpha \hat{x}_s$ for α close enough to one. This is a contradiction, and therefore $x_s \in h_s(p, v, x_{-s}, a)$ implies $x_s \in x_s(p, p \cdot x_s, x_{-s}, a)$.

Step 3: For any $(\chi^*, a^*) \in X^S \times \mathcal{A}$ and for every s, if x_s^* solves (A3') with $a = a^*$, $x_{-s} = x_{-s}^*$ and $\omega_s = p \cdot x_s^*$, and if a^* solves (A2') with $\chi = \chi^*$, then (χ^*, a^*) solves (A1') with $\omega_s = p \cdot x_s^*$. Suppose there exists (χ, a) such that $x_s \in x_s(p, p \cdot x_s, x_{-s}, a)$ for every s, $a \in a(\chi)$ and such that $(\chi, a) \notin \varphi(p, p \otimes \chi)$. Since (χ, a) is feasible in (A1'), $(\chi, a) \notin \varphi(p, p \otimes \chi)$ implies that there exist $(\tilde{\chi}, \tilde{a})$ such that $p \cdot \tilde{x}_s \leqslant p \cdot x_s$ for every s and $v(\tilde{\chi}, \tilde{a}) > v(\chi, a)$. Pick $(\hat{\chi}, \hat{a}) \in \varphi(p, p \otimes \chi)$, so that $v(\hat{\chi}, \hat{a}) \geqslant v(\tilde{\chi}, \tilde{a}) > v(\chi, a)$, and, by lemma 8, $\hat{a} \in a(\hat{\chi})$ and, for every s, $\hat{x}_s \in x_s(p, p \cdot x_s, \hat{x}_{-s}, \hat{a})$. Furthermore, $\hat{x}_s \in x_s(p, p \cdot x_s, x_{-s}, \hat{a})$ by lemma 13, and therefore $\hat{x}_s \in x_s(p, p \cdot x_s, x_{-s}, a)$ by lemma 9. This implies $v(\hat{x}_s, x_{-s}, a) = v(x_s, x_{-s}, a)$, hence $u(\hat{x}_s, a) = u(x_s, a)$ for every s. Therefore, by lemma 7, $f(\hat{x}_s) = f(x_s)$ for every s, hence $a'(\hat{\chi}) = a'(\chi)$ by virtue of lemma 10. Since $a \in a(\chi)$, this implies $a \in a(\hat{\chi})$, whence $v(\hat{\chi}, \hat{a}) = v(\hat{\chi}, a)$. Finally, since $u(\hat{x}_s, a) = u(x_s, a)$ for every s, it follows that $v(\hat{\chi}, a) = v(\chi, a)$ and therefore $v(\hat{\chi}, \hat{a}) = v(\chi, a)$, which contradicts $v(\hat{\chi}, \hat{a}) > v(\chi, a)$. $\square$