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Abstract

We propose and estimate an open economy general equilibrium model that includes international trade between Canada and the US. For both countries, we consider a rich fiscal policy sector with two different types of public expenditure: productive and unproductive government spending. We estimate our model using a new adaptive methodology based on the Mixture of Student's t by Importance Sampling weighted Expectation-Maximization (MitISEM). Our findings regarding the Canadian economy indicate that, irrespective of the type of government expenditure, an increase in domestic public spending leads to an improvement of the domestic trade balance. Notably, we find that the domestic real exchange rate appreciates in response to a positive shock in the domestic unproductive government expenditure, whereas it depreciates after an increase in the domestic productive government spending. Our analysis indicates that a decrease in trade openness, for example resulting from a possible trade war, has important consequences for the propagation of productive and productive government spending shocks on the economy.

Keywords: Open-Economy Model, Fiscal Policy, Adaptive Importance Sampling, Expectation-Maximization.
JEL Classification: E62, F41, C12, C22.

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1 Introduction

International policy-makers, mass media and public opinion have renewed their interest in the interaction between international trade and fiscal policy after the winning comeback election of Donald Trump given his promises of radical policy changes.[1] This is especially true for Canada, in which the trade with the US economy accounts for approximately 70 percent of its overall trade. In this paper, we aim to empirically assess the effects of fiscal policy shocks in open-economy models. This topic has been widely studied in the economic literature. Traditional analysis based on the Mundell-Fleming model predicts that, in a floating exchange rate regime, a higher government expenditure tends to increase aggregate demand and puts upward pressure on interest rates. This generates a capital inflow and an appreciation of the currency that, in turn, crowds out net exports.[2] Despite these predictions, recent literature has found a variety of results in terms of exchange rate and trade balance responses to government spending shocks (see, for example, Backus et al., 1994; Corsetti and Müller, 2006; Born et al., 2013; Corsetti et al., 2012; Auerbach and Gorodnichenko, 2016).

Our paper contributes to these studies in several ways. Firstly, our New Keynesian model features two types of government expenditures, namely, productive and unproductive spending. Secondly, we extend the work by Erceg et al. (2008) and include distortive taxes on capital and labour incomes, together with several fiscal policy rules (see, for example, Leeper et al., 2010a). Our large-scale general equilibrium model features 164 equations with 86 parameters. Traditional estimation techniques based on Random Walk Metropolis Hastings (RWMH) would require a long time to estimate such a model. Accordingly, our third contribution with respect to the previous literature is the proposal of a new estimation strategy, i.e., the MitISEM methodology that can easily tackle complex general equilibrium models. Our model is estimated using quarterly data for Canada and the US for the sample period 1981:Q1-2019:Q1.

The MitISEM methodology proposed in Hoogerheide et al. (2012a) and further developed in Baştürk et al. (2017) was originally applied to time-series modelling. In this paper, we use it in a general equilibrium model. This estimation technique can be summarised as follows. After an initial set of candidate draws, the algorithm applies Importance Sampling (IS) to compute the unknown

[1] Olivier Coibion and Nitya Pandalai-Nayar summarise these promises in their short blog entitled "What does the Trump win mean for the U.S. and global economic outlook?" at `https://sites.utexas.edu/macro/2024/11/13/what-does-the-trump-win-mean-for-the-u-s-and-global-economic-outlook/`.

[2] See, for example, Bryant et al. (1988), Taylor (1993a), Baxter (1995), Kollmann (1998), Betts and Devereux (2000) and McKibbin and Sachs (2011).

posterior density by a mixture of Student's t densities. Importance weights emphasize certain values of the posterior distributions, which achieve identification of the posterior densities. Moreover, at each simulation, the parameters of the mixture of Student's t densities adapt to the most recent IS draws and are recomputed via an Expectation-Maximisation (EM) step. MitISEM presents numerous advantages compared to the RWMH in the estimation of complex general equilibrium models. Firstly, on the computational side, the algorithm is "embarrassingly parallelizable" on multiple processors or graphics processing units (GPU) (see, for example, Baştürk et al., 2016). This allows any user to estimate complex general equilibrium models in a reasonable computing time. Secondly, the algorithm relies on the mixture of the Student's t densities that can easily handle the asymmetry, non-normality and multi-modality of the posteriors. This aspect is particularly important when the likelihood function is complex and not well distributed.

As mentioned above, the recent literature does not agree on the response of trade balance following an increase in public spending. Focusing on the US economy, Corsetti and Müller (2006) show that fiscal expansion has a positive effect on external balance. Kim and Roubini (2008) support these findings for the US economy with a Vector Auto Regression (VAR) model. In particular, these authors find that an expansionary fiscal policy shock induces an improvement in the trade balance. This result has been called "twin divergence", i.e., trade balance and government budget deficit move in opposite directions. Similarly, Müller (2008) shows that a temporary increase in government spending increases net exports. On the other hand, Monacelli and Perotti (2010) which apply VAR techniques to a set of OECD countries find that an increase in government spending results in a trade balance deficit. This is known as "twin deficits", i.e. a positive correlation between budget and current account deficits. In line with this finding, Corsetti et al. (2012) show that in a panel of OECD countries, expansionary fiscal measures worsen the trade balance.

Focusing on the effects of government spending shocks on exchange rates, previous literature has found contrasting results. In line with traditional predictions of the Mundell-Fleming model, using a two-country real business cycle model Backus et al. (1994) find an appreciation of the real exchange rate in response to a positive government spending shock. Erceg et al. (2005) develop an open economy DSGE model for the US and show that a positive shock to government consumption induces an appreciation of the real exchange rate. Similarly, Corsetti et al. (2012) find that an unexpected increase in government spending triggers a short-lived real appreciation, but then the real exchange rate depreciates over time. Auerbach and Gorodnichenko (2016) show that unexpected shocks in government spending lead to an immediate and tangible appreciation of the US dollar.[3]

However, other studies found the opposite results. Kollmann (2010) shows that, if government spending shocks are very persistent and international financial markets incomplete, an increase in government spending may depreciate the real exchange rate. Monacelli and Perotti (2010) use a small open-economy New Keynesian model to show that higher government spending depreciates the real exchange rate. Ravn et al. (2012) achieve the same result assuming that the preferences of private households and the government are characterised by deep habits. Ilzetzki et al. (2013) find that government shocks in high-income countries tend to depreciate the exchange rate in the long run. Using VAR models on the US economy, Enders et al. (2011) find that exogenous expansions of government spending depreciate the real exchange rate. Bouakez et al. (2014) rationalises this finding by developing a small open-economy model that features three key ingredients: incomplete and imperfect international financial markets, sticky prices, and a loose monetary policy.

Focusing on the Canadian economy, our empirical results show that, regardless of the type of government expenditure, an increase in public spending for the domestic economy induces an improvement in the domestic trade balance. This contrasts with the predictions of the Mundell-Fleming model, which ignores the intertemporal investment decisions of economic agents. On the contrary, our model shows that increases in both productive and unproductive government expenditures generate a large crowding-out effect on aggregate investment. Our analysis indicates that the negative effect of an increase in government spending on aggregate investment is due to distortive taxes, the quick adjustment of government transfers, and the persistence of government spending shocks. This drop in investment more than offsets the deterioration of national savings that follows an increase in government spending. Consequently, net exports increase.

Moreover, we find that the response of the real exchange rate after a fiscal expansion depends crucially on the type of expenditure. An increase in unproductive government expenditure induces an appreciation of the domestic real exchange rate. On the other hand, the domestic real exchange rate depreciates in response to an increase in productive government spending. These different responses depend on an important transmission channel that the Mundell-Fleming model misses, i.e., the dynamic interaction between fiscal and monetary policy. This interaction has been studied by several papers that focused mainly on closed economies with non-Ricardian consumers (see, for example, Rigon and Zanetti, 2018; Leith and Von Thadden, 2008). Our model shows that in an

[3] Auerbach and Gorodnichenko (2016) use daily data on US defence spending (announced and actual payments) and find that unexpected shocks to announced military spending, rather than actual outlays on military programmes, lead to an immediate appreciation of the US dollar.

open-economy set up, different degrees of monetary policy activism affect the real interest rate and, in turn, determine the net capital inflow or outflow. In particular, a more (less) aggressive monetary policy implies a higher (lower) real interest rate and thereby induces a net capital inflow (outflow). Accordingly, the supply of domestic currency reduces (improves) and the real exchange rate appreciates (depreciates).

We also perform an analysis to determine output present-value multipliers, for which previous literature has found a wide range of values (see, for example, Auerbach and Gorodnichenko, 2012; Ilzetzki et al., 2013; Ghassibe and Zanetti, 2022; Fernández-Villaverde et al., 2024). With respect to these studies, we are able to evaluate two types of government spending, namely productive and unproductive expenditures. Our results show that output present-value multipliers are higher in response to productive government spending shocks than unproductive government expenditure shocks, this is especially true in the long run. As a robustness exercise, we compute output present-value multipliers for different values of the trade elasticity between Canada and the US. We find that, in the short run, the present-value multipliers for output have the highest values in the case of the model with a low degree of trade elasticity. On the contrary, in the long run, output multipliers are larger in the model with a higher degree of trade openness. This seems to suggest that a possible trade war could be beneficial in the short term but detrimental in the long term.

The remainder of this paper is organised as follows. Section 2 describes the new estimation approach that we use in this paper, i.e., the MitISEM methodology. Section 3 introduces our theoretical model. In Section 4, we present the data used for the analysis, our Bayesian estimates, and we compare the impulse responses for productive and unproductive government spending shocks. In Section 5, we discuss the main transmission mechanisms of government spending shocks with specific interest on the real exchange rate and the trade balance. Section 6 provides the results for output present-value multipliers with a specific focus on the degree of trade openness. Finally, Section 7 concludes the paper.

2 MitISEM for estimation of general equilibrium models

This section details the estimation methodology for our general equilibrium model, which relies on customizing the proposal distribution within an IS algorithm. In particular, we adapt the MitISEM introduced by Hoogerheide et al. (2012b) and improved in Baştürk et al. (2017) to estimate our large-scale general equilibrium model.[4]

2.1 MitISEM estimation algorithm

The algorithm is given by the following steps to obtain an approximation of a target density, i.e., the unknown parameter posteriors. Let us assume that the objective is to estimate a vector of parameters $\boldsymbol{\theta} = (\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(N)})$ from the unknown posterior $p(\boldsymbol{\theta}|Y)$.

1. **Initialization**: Simulate parameters draws $\boldsymbol{\theta}_0 = (\boldsymbol{\theta}_0^{(1)}, \ldots, \boldsymbol{\theta}_0^{(N)})$ from a "naive" Student's t candidate distribution, g_{naive}:

$$g_{naive} \sim t(\mu_0, \Sigma_0, \nu_0), \tag{1}$$

   where μ_0 and Σ_0 are the mean and scale matrix of the Student's t distribution, respectively. They are computed from a preliminary maximization of the log kernel posterior density (equal to log-priors plus log-likelihood) evaluated at the mode. Therefore, the initialization depends on both the prior assumption and the likelihood. The degrees of freedom ν_0 are a-priori chosen by the user. We suggest applying a low value in order to allow for fat tails, for example, $\nu_0 = 3$. Moreover, we apply the same degrees of freedom for all parameters, but this assumption can be relaxed. All the parameters are drawn jointly from g_{naive} and simulations are independent across draws. This step corresponds to an independent Metropolis-Hastings step where the candidate is the g_{naive} distribution and the acceptance rate is 1.

2. **Adaptation**: Estimate the mean (μ^0_{Adap}) and the covariance matrix (Σ^0_{Adap}) of the target distribution by applying an IS method to the draws $\boldsymbol{\theta}_0^{(1)}, \ldots, \boldsymbol{\theta}_0^{(N)}$ from g_{naive} in step 1.[5] IS emphasizes certain areas of the importance distribution (g_{naive}) by sampling more frequently from these values. The importance weights are computed as the ratio between the target distribution and the importance distribution:

$$\omega_{\boldsymbol{\theta}_0} = \frac{p(\boldsymbol{\theta}_0)|Y}{g_{naive}(\boldsymbol{\theta}_0|Y)}. \tag{2}$$

   They are used to generate a new sample of draws $\boldsymbol{\theta}_{0,Adap}^{(1)}, \ldots, \boldsymbol{\theta}_{0,Adap}^{(N)}$ from:

$$g^0_{Adap} \sim t(\mu^0_{Adap}, \Sigma^0_{Adap}, \nu). \tag{3}$$

[4]See the R library in Baştürk et al. (2017) for applications to financial data.
[5]For more details, see Robert and Casella (1999).

Compute the IS weights $\omega_{\boldsymbol{\theta}_{Adap}}$ for this sample. The basic methodology in IS is to choose a distribution which "encourages" the important values.

3. **IS-weighted EM algorithm**: Apply the Expectation-Maximization (EM) algorithm (see Appendix) given the latest IS weights and draws from step 2. The previous draws are used to derive the new candidate density g^h_{Adap} which is a mixture of Student's t densities:

$$g^h_{Adap} = \sum_{h=1}^{H} \eta_h t_h(\mu_h, \Sigma_h, \nu_h), \tag{4}$$

with optimized mean (μ_h), covariance (Σ_h), degrees of freedom (ν_h), and mixture weight (η_h) computed using an EM algorithm on IS weights and draws from step 2. In the first Monte Carlo draw $H = 1$ (there is only one component) and $\eta_h = 1$ (the only component takes all the weights). Draw a new sample $\boldsymbol{\theta}^{(1)}_{h,Adap}, \ldots, \boldsymbol{\theta}^{(N)}_{h,Adap}$ from the distribution that corresponds with this proposal density and compute corresponding IS weights.

4. **Iterate on the number of mixture components**: Given the current mixture of H components with corresponding μ_h, Σ_h, ν_h and η_h, $h = 1, \ldots, H$, take a percentage of the sample $\boldsymbol{\theta}^{(1)}_{h,Adap}, \ldots, \boldsymbol{\theta}^{(N)}_{h,Adap}$ that corresponds to the highest IS weights. Construct a new mode μ_{H+1} and scale matrix Σ_{H+1} with these draws and IS weights, which are the starting values for the additional component in the mixture candidate density in equation (4). This step ensures that the new component covers a region of the parameter space in which the previous candidate mixture had a relatively low probability mass. Usually, two or three components are sufficient given the flexibility of the mixture of Student's t densities. In the case that the maximum number of components chosen a priori is reached or the convergence in step 5 is achieved, the drawing of a new component is skipped. Given the latest IS weights and the draws from the current mixture of H components, apply the EM algorithm to optimize (again) each mixture component μ_h, Σ_h, ν_h and η_h with $h = 1, \ldots, H+1$. Draw a new sample from the mixture of $H + 1$ components and compute the corresponding IS weights.

5. **Assess convergence of the candidate density quality by inspecting the IS weights** and return to step 3 unless the algorithm has converged.

Initial values are usually obtained through grid-search algorithms that may incur in local maxima and not positive-definite Hessians. In such cases, the user can specify a reasonable starting value for μ_0 and Σ_0 that will be updated in the adaptation step. This eliminates the strong dependence of the results on the values specified by the user and enhances robustness. Additionally, to prevent a particularly poor approximation at the first iteration, especially in high-dimensional settings, it is advisable to combine the prior distribution with an approximation at the mode.

Step 2 can be seen as an intermediate step that quickly attempts to improve the initial candidate density g_{naive}. If during the EM algorithm, a scale matrix Σ_h of a Student's t component becomes (nearly) singular, then this h-th component is removed from the mixture. Moreover, if during the EM algorithm, a weight η_h becomes very small, then this h-th component is removed from the mixture.

The convergence in step 4 can be assessed by computing the relative change in the Coefficient of Variation (CoV) of the IS weights, that is, the standard deviation of the IS weights divided by their mean (see, for example, Hoogerheide et al., 2012b). The default convergence in MitISEM is defined as the change of the CoV being smaller than 2 percent. The convergence tolerance can also be changed by the user. Finally, the starting values specified for ν_{H+1} and η_{H+1} in step 4 are fixed to 1 and 0.10, i.e., the new component has fat tails and a relatively low probability ex ante.

3 Theoretical model

Our theoretical framework encompasses international trade between two countries, namely Canada and the US. We assume that these two countries differ in size but are otherwise symmetric.

In each country, the representative household maximizes its utility function that has two arguments, consumption and labour. The representative household makes investment decisions, owns the capital stock, and rents it to intermediate production firms. Each country produces a single final production good and a continuum of intermediate production goods. As in Leeper et al. (2010b), intermediate production firms produce under monopolistic competition and use three input factors, i.e., private capital, government capital, and labour. Nominal rigidities in each country consist of sticky prices and wages à la Calvo (1983), as well as partial indexation of both wages and prices to past inflation rates. In each country, final consumption goods, as well as investment goods, are produced by firms that combine domestic and imported goods under perfect competition. As in Bodenstein et al. (2011), we assume that asset markets are complete at the country level but are incomplete internationally. In both countries, we assume a rich fiscal sector that includes different types of government expenditure, namely productive and unproductive government spending. In addition, we consider several fiscal policy rules. Since the theoretical framework is symmetric, in the following we describe only the model for the domestic country, which is Canada, in our study.

3.1 Households

The representative household maximizes its lifetime utility function by choosing purchases of consumption ($C_{1,t}$), and investment goods ($I_{1,t}$), capital stock ($K_{1,t}$), and next period's holdings of both domestic government bonds ($B_{1,t+1}$), and foreign government bonds ($B^f_{1,t+1}$), given its period-by-period budget constraint. Therefore, the representative household maximizes:

$$\max_{\left\{C_{1,t},I_{1,t},K_{1,t},B_{1,t},B^f_{1,t}\right\}} \mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta_1^t \left[\frac{1}{1-\sigma_1^c} \left(C_{1,t} - h_1 C_{1,t-1}\right)^{1-\sigma_1^c} \exp\left(\frac{\sigma_1^c - 1}{1+\sigma_1^l} \left(L_{1,t}\right)^{1+\sigma_1^l} \right) \right] \right\}, \quad (5)$$

subject to the budget constraint:

$$\begin{aligned} & P^c_{1,t} C_{1,t} + P^i_{1,t} I_{1,t} + \left(R^b_{1,t}\right)^{-1} B_{1,t+1} + \frac{e_{1,t} \left(R^b_{2,t}\right)^{-1} B^f_{1,t+1}}{\phi^b_{1,t}} \quad (6) \\ & = B_{1,t} + e_{1,t} B^f_{1,t} + (1-\tau^l_{1,t}) W_{1,t} L_{1,t} + (1-\tau^k_{1,t}) R^k_{1,t} K_{1,t-1} + D_{1,t} + T_{1,t}, \end{aligned}$$

and the capital accumulation equation:

$$K_{1,t} = \left(1-\delta_1\right) K_{1,t-1} + \varepsilon^i_{1,t} \left(1 - S \left(\frac{I_{1,t}}{I_{1,t-1}} \right)^2 \right) I_{1,t}. \quad (7)$$

In equation (5), $\mathbb{E}_t$ denotes the expectation operator at time t and β_1^t is the discount factor. Representative household consumption is influenced by the presence of an external habit (h_1) related to aggregate past consumption. The parameter σ_1^c is the coefficient of relative risk aversion. The variable $L_{1,t}$ represents the hours worked, while σ_1^l is the inverse of the elasticity of work with respect to the real wage.

In equation (6), $P^c_{1,t}$ and $P^i_{1,t}$ indicate the prices of consumption and investment goods, respectively. The gross nominal return of the domestic government bond is denoted by $R^b_{1,t}$, while $R^b_{2,t}$ is the gross nominal return of the foreign government bond. The latter is denominated in foreign currency and therefore its domestic value depends on the nominal exchange rate ($e_{1,t}$) expressed in units of domestic currency per unit of foreign currency. As in the paper of Erceg et al. (2008), we assume that the representative household faces an intermediation cost to purchase the foreign bond, $\phi^b_{1,t}$. We indicate by $W_{1,t}$ the aggregate nominal wage, while $R^k_{1,t}$ is the rental rate for capital services. $D_{1,t}$ represents the dividends paid by the production goods firms that are owned by the representative household. Moreover, the fiscal authority absorbs part of the gross income of the representative household to finance its expenditure. Accordingly, in equation (6), $\tau^{l}_{1,t}$ denotes the labour income tax rate, while $\tau^{k}_{1,t}$ is the capital income tax rate. Moreover, $T_{1,t}$ indicates lump-sum transfers from the government.

The capital accumulation equation (7) includes the adjustment cost function $S(\cdot)$ and an investment-specific technology shock denoted by $\varepsilon^{i}_{1,t}$. Finally, δ_1 denotes the depreciation rate. We also assume that the representative household has monopoly power over wages that implies sticky nominal wages à la Calvo (1983). Finally, we allow for a partial indexation of wages to past inflation rates.

3.2 Firms: production of consumption goods

The final consumption good ($C_{1,t}$) is produced under perfect competition and sold to the representative household. The representative firm producing final consumption goods uses a constant elasticity of substitution production function. In particular, domestic ($C^{d}_{1,t}$) and foreign ($M^{c}_{1,t}$), intermediate consumption goods are combined to obtain final consumption goods. The cost minimization problem faced by the representative firm producing final consumption goods is given by:

$$\min_{\left\{C^{d}_{1,t}, M^{c}_{1,t}\right\}} P^{d}_{1,t}C^{d}_{1,t} + P^{m}_{1,t}M^{c}_{1,t},$$

$$s.t.: C_{1,t} = \left(\left(\omega^{c}_{1}\right)^{\frac{\rho^{c}_{1}}{1+\rho^{c}_{1}}} \left(C^{d}_{1,t}\right)^{\frac{1}{1+\rho^{c}_{1}}} + \left(\omega^{mc}_{1}\right)^{\frac{\rho^{c}_{1}}{1+\rho^{c}_{1}}} \left(\varepsilon^{m}_{1,t}M^{c}_{1,t}\right)^{\frac{1}{1+\rho^{c}_{1}}} \right)^{1+\rho^{c}_{1}}. \tag{8}$$

We denote by ω^{c}_{1} and ω^{mc}_{1} the weights of domestic and foreign consumption goods. Moreover, ρ^{c}_{1} represents the elasticity of substitution between domestic and foreign intermediate goods. We assume that import preferences are driven by an exogenous shock, $\varepsilon^{m}_{1,t}$, which has the form of a AR(1) process. The Lagrange multiplier associated with the cost minimization problem of the representative firm producing final consumption goods is defined as the price of consumption goods ($P^{c}_{1,t}$).

3.3 Firms: production of investment goods

Firms producing investment goods $(I_{1,t})$ use a nested constant elasticity of substitution production function. These firms operate under perfect competition and sell investment goods to the representative household. In particular, domestic and foreign investment goods, denoted respectively by $I^d_{1,t}$ and $M^i_{1,t}$, are combined to obtain final investment goods. We can express the cost minimization problem of a typical firm producing investment goods as follows:

$$\min_{\left\{I^d_{1,t},M^i_{1,t}\right\}} P^d_{1,t}I^d_{1,t} + P^m_{1,t}M^i_{1,t},$$
$$s.t : I_{1,t} = \left(\left(\omega^i_1\right)^{\frac{\rho^i_1}{1+\rho^i_1}}\left(I^d_{1,t}\right)^{\frac{1}{1+\rho^i_1}} + \left(\omega^{mi}_1\right)^{\frac{\rho^i_1}{1+\rho^i_1}}\left(\varepsilon^m_{1,t}M^i_{1,t}\right)^{\frac{1}{1+\rho^i_1}}\right)^{1+\rho^i_1}, \tag{9}$$

where ω^i_1 and ω^{mi}_1 indicate the weights of domestic and foreign investment goods. The elasticity of substitution between domestic and foreign goods is denoted by ρ^i_1. We also assume that investment goods are influenced by an import preferences shock, $\varepsilon^m_{1,t}$, that is the same we assumed in the production of consumption goods. The Lagrange multiplier associated with the problem of cost minimization of the typical investment goods firm coincides with the price of investment goods $P^i_{1,t}$.

3.4 Firms: production of domestic intermediate goods

Each country produces a single final production good and a continuum of intermediate production goods. Each intermediate good firm j produces its differentiated output using the Cobb-Douglas technology with three input factors, i.e., private capital $(K_{1,t})$, labour $(L_{1,t})$ and productive government capital $(K^{gp}_{1,t})$:

$$\min_{\left\{K_{1,t}(j),K^{gp}_{1,t}(j),L_{1,t}(j)\right\}} \left(R^k_{1,t}K_{1,t}(j) + W_{1,t}L_{1,t}(j) + P^{kg}_{1,t}K^{gp}_{1,t}(j)\right),$$
$$s.t. : Y_{1,t}(j) = \varepsilon^a_{1,t}\left(K_{1,t}(j)\right)^{\alpha^k_1}\left(L_{1,t}(j)\right)^{\alpha^l_1}\left(K^{gp}_{1,t}(j)\right)^{\alpha^{kg}_1}, \tag{10}$$
$$\text{where : } \alpha^k_1 + \alpha^l_1 = 1 \text{ and: } 0 < \alpha^{kg}_1 < 1,$$

where α^k_1 and α^l_1 indicate the private capital and labour share in production, respectively. Equation (10) displays an additional parameter associated with the productive government capital, that is α^{kg}_1. This parameter denotes the public capital share in production. Moreover, $\varepsilon^a_{1,t}$ indicates the total factor productivity exogenous shock following a first order autoregressive process. Firms set their prices according to current and expected marginal costs, but also according to the past inflation rate. In our case, the marginal cost does not only depend on wages and the capital rental rate, but also on the price of the productive government capital. In this regard, we assume that the evolution equation for productive government capital is given by:

$$K^{gp}_{1,t+1}\left(j\right) = \left(1-\delta^g_1\right) K^{gp}_{1,t}\left(j\right) + GP^{d}_{1,t}, \tag{11}$$

where δ^g_1 is the parameter indicating the depreciation rate of the productive government capital. Moreover, $GP^{d}_{1,t}$ indicates the domestic productive government investment.

We also assume that intermediate production firms set prices according to the Calvo (1983) model. As an additional assumption concerning nominal rigidities, we allow for partial indexation of both wages and prices to past inflation rates.

3.5 Fiscal authority

The government finances its public spending by issuing bonds or adjusting taxes and transfers. We separate domestic government spending into unproductive ($GU^d_{1,t}$) and productive ($GP^d_{1,t}$) expenditures. Therefore, the fiscal authority's period-by-period budget constraint has the following form:

$$P^{gu}_{1,t} GU^d_{1,t} + P^{gp}_{1,t} GP^d_{1,t} + B_{1,t} + T_{1,t} = \tau^r_{1,t} + \left(R^b_{1,t}\right)^{-1} B_{1,t+1},$$

where $\tau^r_{1,t}$ denotes the total government distortionary tax revenues that are given by:

$$\tau^r_{1,t} = \tau^l_{1,t} W_{1,t} L_{1,t} + \tau^k_{1,t} R^k_{1,t} K_{1,t-1},$$

In line with Leeper et al. (2010a), we assume that the log-linearized expressions for the fiscal policy rules are:

$$\hat{\tau}^l_{1,t} = \phi^{yl}_1 \hat{y}^d_{1,t} + \gamma^{bl}_1 \hat{b}_{1,t-1} + \hat{\varepsilon}^l_{1,t}, \tag{12}$$

$$\text{where}: \hat{\varepsilon}^l_{1,t} = \rho^l_1 \hat{\varepsilon}^l_{1,t-1} + \eta^l_{1,t}, \tag{13}$$

$$\hat{\tau}^k_{1,t} = \phi^{yk}_1 \hat{y}^d_{1,t} + \gamma^{bk}_1 \hat{b}_{1,t-1} + \hat{\varepsilon}^k_{1,t}, \tag{14}$$

$$\text{where}: \hat{\varepsilon}^k_{1,t} = \rho^k_1 \hat{\varepsilon}^k_{1,t-1} + \eta^k_{1,t}, \tag{15}$$

$$\hat{t}_{1,t} = -\phi^{yt}_1 \hat{y}^d_{1,t} - \gamma^{bt}_1 \hat{b}_{1,t-1} + \hat{\varepsilon}^t_{1,t}, \tag{16}$$

$$\text{where}: \hat{\varepsilon}^t_{1,t} = \rho^t_1 \hat{\varepsilon}^t_{1,t-1} + \eta^t_{1,t}, \tag{17}$$

$$\hat{gp}^d_{1,t} = \phi^{ygp}_1 \hat{y}^d_{1,t} - \gamma^{bgp}_1 \hat{b}_{1,t-1} + \hat{\varepsilon}^{gp}_{1,t}, \tag{18}$$

$$\text{where}: \hat{\varepsilon}^{gp}_{1,t} = \rho^{gp}_1 \hat{\varepsilon}^{gp}_{1,t-1} + \eta^{gp}_{1,t}, \tag{19}$$

$$\hat{gu}^d_{1,t} = -\phi^{ygu}_1 \hat{y}_t - \gamma^{bgu}_1 \hat{b}_{1,t-1} + \hat{\varepsilon}^{gu}_{1,t}, \tag{20}$$

$$\text{where}: \hat{\varepsilon}^{gu}_{1,t} = \rho^{gu}_1 \hat{\varepsilon}^{gu}_{1,t-1} + \eta^{gu}_{1,t}, \tag{21}$$

where the small letters with the hats denote log-deviations of the variables from their respective steady states. Moreover, we assume that all the coefficients in the fiscal rules have positive values, that is, $\phi^x_1 \geq 0$ for $x = \{yl, yk, yt, ygp, ygu\}$ and $\gamma^z_1 \geq 0$ for $z = \{bl, bk, bt, bgp, bgu\}$. Fiscal rules (12), (14), (16), (18) and (20) imply that fiscal variables respond to contemporaneous changes in output and with a delay of one quarter to variations in government debt. Moreover, equations (12), (14), (16), (18) and (20) include five distinct exogenous AR(1) processes $\hat{\varepsilon}^l_{1,t}$, $\hat{\varepsilon}^k_{1,t}$, $\hat{\varepsilon}^t_{1,t}$, $\hat{\varepsilon}^{gp}_{1,t}$ and $\hat{\varepsilon}^{gu}_{1,t}$, with $\rho^v_1 \in [0,1]$ for $v = \{l, k, t, gp, gu\}$ and all η_1's are distributed i.i.d. $\mathcal{N}(0,1)$. Finally, we assume that productive spending responds positively to increases in aggregate output (see, for example, Ambler et al., 2017), whereas unproductive government spending responds negatively (see, for example, Leeper et al., 2010a).

3.6 Central bank

The central bank is assumed to follow a Taylor rule (Taylor, 1993b) specified in terms of the past nominal interest rate, domestic inflation and output gap:[6]

$$\frac{R^b_{1,t}}{\left(R^b_1\right)^{SS}} = \left(\frac{R^b_{1,t-1}}{\left(R^b_1\right)^{SS}}\right)^{\rho^r_1} \left[\left(\frac{\pi^d_{1,t}}{\left(\pi^d_1\right)^{SS}}\right)^{r^\pi_1} \left(\frac{Y^d_{1,t}}{Y^{dp}_{1,t}}\right)^{r^y}\right]^{(1-\rho^r_1)} \left(\frac{Y^d_{1,t}/Y^d_{1,t-1}}{Y^{dp}_{1,t}/Y^{dp}_{1,t-1}}\right)^{r^{\Delta y}_1} \varepsilon^r_{1,t}, \tag{22}$$

where $\left(R^b_1\right)^{SS}$ and $\left(\pi^d_1\right)^{SS}$ indicate the steady-state values for the nominal interest rate and domestic inflation, respectively. Moreover, we denote by ρ^r_1 the interest rate smoothing parameter, while r^y denotes the response of the nominal interest rate to the output gap, $r^{\Delta y}_1$ indicates the response of the nominal interest rate to changes in the output gap, and r^π_1 represents the reaction of the interest rate on domestic inflation. We denote by $\varepsilon^r_{1,t}$ the monetary policy shock that follows a AR(1) process.

3.7 Market clearing condition

Imposing the market-clearing condition for the good market of the domestic economy implies the following aggregate resource constraint:

$$Y^d_{1,t} = C^d_{1,t} + I^d_{1,t} + GU^d_{1,t} + GP^d_{1,t} + \frac{\zeta_2}{\zeta_1} M_{2,t}, \tag{23}$$

$$\text{where} : M_{2,t} = M^c_{2,t} + M^i_{2,t},$$

where $M_{2,t}$ indicates the net imports of the foreign country, while ζ_1 and ζ_2 represent the relative population sizes of the home and foreign country, respectively. Simply, the market clearing condition (23) states that the production of domestic firms is equal to the domestic demand of the representative household for consumption and investment goods, plus domestic government expenditure and total imports from the foreign country.

3.8 Bilateral relations

For country 1, the relative import prices can be expressed as follows:

$$\frac{P^m_{1,t}}{P^d_{1,t}} = \frac{e_{1,t}P^c_{2,t}}{P^c_{1,t}} \frac{P^d_{2,t}}{P^c_{2,t}} \frac{P^c_{1,t}}{P^d_{1,t}}, \tag{24}$$

[6] We define the output gap as the difference between actual ($Y^d_{1,t}$) and potential output ($Y^{dp}_{1,t}$).

where $P_{1,t}^m$ is the price of imported goods, whereas $P_{1,t}^d$ indicates the price of the final production good. Moreover, the consumption real exchange rate is given by:

$$rer_{1,t} = \frac{e_{1,t} P_{2,t}^c}{P_{1,t}^c}. \tag{25}$$

We assume that the domestic holdings of internationally traded bonds (that is, the home country's net foreign assets, denominated in foreign currency) evolve according to:

$$\frac{e_{1,t} \left(R_{2,t}^b \right)^{-1} B_{1,t+1}^f}{\phi_{1,t}^b} = e_{1,t} B_{1,t}^f + \frac{\zeta_2}{\zeta_1} e_{1,t} P_{2,t}^m \left(M_{2,t}^c + M_{2,t}^i \right) - P_{1,t}^m \left(M_{1,t}^c + M_{1,t}^i \right), \tag{26}$$

where $M_{2,t}^c$ and $M_{2,t}^i$ indicate the foreign country imports of consumption and investment goods, respectively. Finally, the market clearing condition for foreign asset holdings is $B_{1,t}^f + B_{2,t}^f = 0$.

4 Estimation results

In this section, we describe the data used to evaluate the theoretical model. Then, we discuss the priors of the endogenous parameters, and the exogenous processes related to the structural shocks. Finally, we present the main estimation results.

4.1 Data

We estimate the model using data for Canada and the US for the sample period 1981:Q1-2019:Q1. As mentioned above, trade between Canada and the US accounts for approximately 70 percent of total Canadian trade. This implies that trade with the US provides a realistic characterisation of the rest of the world for Canada.

Since there are twenty-two exogenous shocks in the model, twenty-two data series are used in the estimation, we plot them in Figure 1. We use data on Canadian and US for real gross domestic products, real private investments, real wage compensations, inflation rates, nominal interest rates, real labour tax revenues, real capital tax revenues, real productive government expenditures, real unproductive government expenditures, real government lump-sum transfers, and Canadian real imports from the US and US real imports from Canada.

We use the OECD Economic Outlook database[7] for most of our variables. The only exceptions

[7] We use the OECD Economic Outlook database no. 106, available at `https://www.oecd.org/en/publications/oecd-economic-outlook_16097408.html`.

are the series for the wage compensations, the nominal interest rates and the imports. The series for the wage compensation in Canada is taken from Statistics Canada. The series for the wage compensation in the US is taken from the US FRED. The series of the Canadian nominal interest rate is constructed using data from the IMF and the Bank of Canada. The series of the US nominal interest is taken from the US FRED. The series of imports for both countries are taken from the IMF (Direction of Trade Statistics).

For each country, to obtain the real variables, we deflate the nominal variables using the country-specific GDP deflator.[8] Then the real variables are converted into per capita terms by dividing for the country-specific working-age population. Following Leeper et al. (2010a) and Pfeifer (2014), we detrend the logarithm of each real variable separately,[9] while we demean the inflation rate and nominal interest rate.[10]

Following Asimakopoulos et al. (2021), we assume that productive government spending includes expenditures with a substantial (physical or human) capital component, whereas the unproductive spending category relates to government final wage and non-wage consumption expenditures. Accordingly, government productive expenditure is composed of government fixed capital formation, capital payments, and government consumption of fixed capital. Unproductive government spending corresponds to government final consumption expenditure. We also assume that the series of government transfers is given by the social security benefits paid by the government. The detailed description of data construction and sources for the observed variables of the model is reported in online Appendix B.

4.2 Model parameters

We choose to divide the parameters into three different sets: the first corresponds to parameters that are kept fixed and set according to the previous economic literature; the second set is constructed from the observed data; and the third set is estimated with MitISEM.

[8] The only exception is the series of Canadian imports from the US. Since the original series is expressed in US dollars, we use the US GDP deflator to deflate this series.

[9] In particular, we use the HP filter with a smoothing parameter equal to 1,600.

[10] Some studies (see, for example, Greenwood et al., 1997, Greenwood et al., 2000, Altig et al., 2011, Schmitt-Grohé and Uribe, 2012) have estimated DSGE models including one or two common stochastic trends. This strategy is feasible when the number of trends is limited to one or two, but it becomes non-trivial in the presence of a larger number of trends. In this regard, Leeper et al. (2010a) argued that in models that analyse fiscal policy, the number of trends is often larger than two because several fiscal variables display their own trends. Moreover, some of these variables, such as transfers, show upward trends, and this requires specific modelling assumptions to guarantee fiscal sustainability. Accordingly, as an estimation strategy, we prefer to follow the treatment of observed variables used by Leeper et al. (2010a).

Fixed and calibrated parameters according to actual data. Table 1 presents the first set of parameters which can be viewed as strict priors because they can be directly related to the steady-state values and are not identifiable from the data we use. To set the values of these parameters, we follow the most recent DSGE literature. Moreover, we assume that these parameters have the same values for both domestic and foreign countries, with the subscript i, indicating $i = \{Canada, US\}$.

We fix the discount factor (β_i) according to Del Negro and Schorfheide (2008). As is common in the literature, we assume a private capital depreciation rate (δ_i) that implies an annual depreciation on capital of 0.10. We assume that the intertemporal elasticity of substitution ($\frac{1}{\sigma_i^c}$) corresponds to a coefficient of relative risk aversion equal to 5, a value frequently employed in macroeconomic studies (see, for instance, Jermann, 1998).

We set the elasticity of the labour supply (σ_i^l) equal to 4 (see Chetty et al., 2013). As in Smets and Wouters (2007), the steady-state mark-up in the labour market (ν_i^w) is equal to 1.50, and we assume that the steady-state mark-up in the goods market (ν_i^p) is also equal to 1.50. Moreover, the curvature parameters of the Kimball aggregators in the goods, (ϑ_i^p), and labour, (ϑ_i^w), market are both set at 10. We follow Bodenstein et al. (2011) and assume a value of 0.0001 for the parameter that captures the curvature of the bond intermediation cost (ϕ_i^b). As in Leeper et al. (2010a), we assume that the depreciation rate for government capital expenditure (δ_i^g) corresponds to 0.005. Moreover, we assume a value of the private capital share in the production function (α_i^k) in line with Leeper et al. (2010b). We set the parameter that indicates the public capital share in the production function (α_i^{kg}), which is in line with the estimates of Asimakopoulos et al. (2021).

In Table 2, we report the second set of parameters derived from the observed data for Canada and the US. Once these parameters are computed, we hold them as fixed to estimate the model. For both countries, the relative shares of productive and unproductive government expenditures on GDP are computed as average ratios for the 1981-2019 period. Similarly, in each country, the steady-state tax rates for capital and labour are obtained from average capital and labour income tax rates, respectively, taken from our sample data. In each country, the share of transfers over GDP has been computed residually from the government budget constraint using the steady states reported above and the relative steady state of the debt-to-output ratio, which is the average annual debt-to-output ratio for the period under consideration.[11]

Between 1981 and 2019, Canadian imports of goods and services from the US accounted for

[11] During the period 1981-2019, the average shares of annual debt over output in Canada and the US are approximately 55 percent and 66 percent, respectively.

approximately 19 percent of Canadian GDP. During the same period, Canadian total imports were divided into 82 percent consumption goods and 18 percent services approximately. On the other hand, US imports of goods and services from Canada accounted for approximately 2 percent of US GDP. US total imports were divided into 83 percent consumption goods and 17 percent services approximately. By combining these statistics, we are able to compute the steady-state parameters and determine trade flows for Canada and the US: the parameters measuring the weight on imports in consumption (ω_1^{mc} and ω_2^{mc}, respectively), as well as the parameters capturing the weight on imports in investment (ω_1^{mi} and ω_2^{mi}, respectively).[12] Finally, Table 2 indicates that the Canadian population accounts for approximately 10 percent of the combined population of the two countries.

Prior distributions. Table 3 shows the third group that includes endogenous parameters for both Canada and the US and is estimated with the MitISEM. We choose priors that are the same for both countries and are in line with previous literature. More specifically, we set the priors for habit in consumption and Calvo probabilities for both wages and prices as in Del Negro and Schorfheide (2008). Our assumption on the prior for investment adjustment costs follows Schmitt-Grohé and Uribe (2012), moreover, we set the priors for indexation parameters of both wages and prices as in Cacciatore and Traum (2022).

Turning to the monetary policy rule, the priors for the degree of interest rate smoothing and for the long-run reaction coefficients of inflation and output are in line with those used by Del Negro and Schorfheide (2008). Following Asimakopoulos et al. (2021), we set the prior of the short-run coefficient of output as Gamma ($\mathcal{G}$) distributed with mean equal to 1.20 and standard deviation of 0.05.

Focusing on the coefficients of the fiscal sector, our priors are in line with Asimakopoulos et al. (2021). Specifically, the priors for the parameters of lump-sum transfers and labour tax rate elasticities with respect to output are assumed to have $\mathcal{G}$ distributions with mean of 0.10 and standard deviation of 0.05. In addition, we assume that the prior for the parameter of the capital tax rate elasticity with respect to output is $\mathcal{G}$ distributed with mean 0.40 and standard deviation 0.20. Our prior distributions for the responses of labour income tax, capital tax, and lump-sum transfers to government debt range approximately between 0 and 0.25, between 0 and 0.75, and between 0 and 1, respectively. We assume that the parameters that measure the responses of productive and unproductive government expenditures to output have a $\mathcal{G}$ distribution with mean

[12] See Online Appendix A for the full derivation of composite parameters.

of 0.15 and standard deviation of 0.05. Moreover, the priors for the parameters that indicate the responses of productive and unproductive government expenditures to debt are $\mathcal{G}$ distributed with mean 0.40 and standard deviation 0.20.

The last row of Table 3, reports the prior for the elasticity of substitution between domestic and foreign goods. Following Cacciatore and Traum (2022), we assume a $\mathcal{G}$ prior with a mean equal to 1.10 and a standard deviation equal to 0.10. Table 4 shows the priors of the parameters related to all exogenous processes in our model. Following Del Negro and Schorfheide (2008), we use Beta ($\mathcal{B}$) distributions for the persistence parameters of the several shocks with prior mean values of 0.75 and prior standard deviations of 0.15. Finally, we use Inverse Gamma ($\mathcal{IG}$) distributions for standard errors of exogenous shocks with mean equal to 0.10 and standard deviation equal to 2.00, as in Herbst and Schorfheide (2014).

4.3 Posterior estimates

Table 3 shows the posterior mean estimates for endogenous parameters with credible intervals at $5th$ and $95th$ percentiles. In general, the parameters are strongly identified with tight posterior distributions.[13]

The posterior means of consumption habit for Canada (h_1) and the US (h_2) correspond to 0.38 and 0.20, respectively. Our results are close to those of previous microeconometric studies that evaluated the presence of habit formation. In this regard, Fuhrer (2000) has unveiled a puzzle arising from the different estimates of habit formation among microeconometric and macroecometric studies. The former papers generally tend to find a much lower degree of habit formation than the latter studies.[14] Therefore, the estimated values of the habit formation parameters obtained from our macroeconomic model seem to reconcile with previous microeconometric literature. Focusing on the investment adjustment cost, our estimated values of φ_1^i for Canada and the US are in line with the values found by Cacciatore and Traum (2022).

In terms of nominal rigidities, our estimates indicate that wages and prices are sticky in Canada and the US. The effects of government spending shocks on private consumption will crucially depend on this result, see Section 4.4.

The posterior mean estimates of the Calvo wage setting probabilities for Canada (ξ_1^w) and the US (ξ_2^w) are higher than our assumed priors. Our results show that the probability of optimally

[13]In online Appendix C, we show all the prior and posterior density functions for the estimated parameters.

[14]For example, Dynan (2000) found no evidence of habit formation for the US economy.

resetting nominal wages in Canada is about 0.10 and in the US is approximately 0.13. Similarly, the estimated means of the Calvo price setting probabilities for Canada (ξ_1^p) and the US (ξ_2^p) are higher than their priors. This implies that the Calvo readjustment probability for Canada is about 0.18, whereas for the US it is approximately 0.13.[15] Turning to the posterior estimates of wage and price indexations, the mean values of Canada are higher than those found by Cacciatore and Traum (2022), whereas those for the US fall within the ranges of values found by Smets and Wouters (2007) and Del Negro and Schorfheide (2008).

Our estimates of the posterior means of the reaction coefficients to inflation for Canada and the US are lower than in Cacciatore and Traum (2022). Whether the Canadian real exchange rate appreciates or depreciates after government spending shocks crucially depends on the reaction of its central bank to the increase in inflation, see Section 4.4. In the long run, both countries exhibit a weaker response of the nominal interest rate to the output gap compared to the short run. Additionally, the posterior of the degree of interest rate smoothing is higher in the US than in Canada.

Regarding the posterior estimates of the fiscal rule parameters, in both countries, we observe that the capital tax response is more procyclical than the labour tax response. For both Canada and the US, capital tax responds more strongly than labour tax to changes in government debt.[16] These distortive taxes induce a strong crowding out effect on private investment following the government spending shock, see Section 4.4.

In addition, our estimates indicate that, in both Canada and the US, lump-sum transfers respond more strongly to changes in the debt-to-output ratio than to output deviations. This result implies that non-distortionary taxation is the preferred option to stabilise debt in both countries. The higher are the values for ϕ_1^{yt} and γ_1^{bt}, the stronger is the crowding-out effect on private investment after a government spending shock, see Section 4.4.

Focusing on the two different types of government expenditure, we observe important differences between Canada and the US. In particular, our results show that in Canada, the productive government spending has a stronger response to changes in output than unproductive expenditure (0.42 and 0.15, respectively). The opposite result is found for the US. Moreover, in Canada, productive government expenditure responds less strongly than unproductive government spending to

[15] Our estimated values of ξ_1^w and ξ_1^p are higher than the values found by Cacciatore and Traum (2022) for Canada, whereas our estimated posteriors for ξ_2^w and ξ_2^p are in line with those found by Del Negro and Schorfheide (2008) for the US.

[16] Our estimated results are in line with many studies in the optimal fiscal policy literature for the US economy (see, for example, Barro, 1979; Chari et al., 1994; Angelopoulos et al., 2015).

debt variations (0.02 and 0.43, respectively). In contrast, our posterior estimates for the US show that γ_2^{bgp} is higher than γ_2^{bgu}.

The posterior estimates of the elasticity between domestic and foreign goods for both Canada and the US are very close to unity. Accordingly, our findings are consistent with the values estimated by Cacciatore and Traum (2022).[17]

In term of exogenous shocks, Table 4 shows the estimated posteriors for the autocorrelation coefficients and standard errors of all exogenous processes, together with their credible intervals at $5th$ and $95th$ percentiles. In general, all exogenous disturbances seem to be well identified with tight posterior distributions. Focusing on $AR(1)$ processes, the shocks to all taxes are less persistent in Canada than in the US. Importantly, for both countries, we observe that the persistence of unproductive spending is much stronger than that of productive spending. This has an important implication in terms of the trade balance response to the two different types of government spending shocks, see Section 4.4. Notable differences in estimated persistence also relate to investment and productivity shocks, with the former higher in the US, whereas the latter is higher in Canada.

Finally, our posterior estimates show that Canada and the US have similar estimated volatilities[18] Focusing on Canada, we note that the shocks of productivity and import preferences are the most volatile. Focusing on the US, we observe that the shocks of monetary policy, productivity, investment, and wage mark-up are more volatile than the remaining shocks.[19]

4.4 Impulse response analysis

Now we focus on the effects of government spending shocks on the main macroeconomic aggregates for our estimated model. In particular, we analyse the IRFs related to productive and unproductive government spending shocks for the home country, i.e., Canada. The lines displayed in the various graphs are generated by the mean estimates of the posterior distributions. In each figure, we show the impulse responses following a 1 percent exogenous positive shock to domestic productive and unproductive government spending.[20] In Figures 2 and 3, we report the IRFs of the model with

[17]The values of our estimated parameters are slightly lower than those found by Cacciatore and Traum (2022) for the substitutability of home and foreign between Canada and the US.

[18]The exception relates to the monetary policy shock that is estimated to be much more volatile for the US economy.

[19]In Online Appendix E, we show the historical decomposition for the Canadian GDP. This figure gives a quarter-by-quarter breakdown of the importance of the different shocks.

[20]Qualitatively the results of the IRFs are the same if we use the estimated standard deviation of the shocks instead of the simulated 1 percent standard deviation. We simply normalise the shock to the economy to be 1 percent to ease the comparison of the impulse responses between the two cases of domestic productive and unproductive government spending. In online Appendix D, we present the estimated impulse responses together with the confidence intervals.

nominal rigidities.[21]

Productive government expenditure. Figure 2 shows that both distortive taxes have positive responses to the shock and, in particular, the capital tax has a larger increase. As we will explain below, this has important consequences on the response of private investment and, in turn, it affects the trade balance. Moreover, the increase in both distortive taxes and the decline in government transfers are not enough to fund the exogenous increase in public spending and, as a consequence, public debt increases substantially.

We observe that a positive shock to domestic productive government spending leads to a consistent increase in domestic output. Moreover, hours worked rise in response to this shock. We also note that aggregate consumption increases. This is due to the increase in the consumption of both domestic and foreign goods. The consumption of foreign goods rises because these goods are cheaper relative to domestic ones. The positive response of private domestic consumption depends on the increase in the domestic wage rate. The intuition behind this result is the following: an increase in productive government spending leads to a rise in aggregate demand, as well as an increase in the marginal product of labour. In turn, this implies that labour demand rises. At the same time, higher future distortionary taxes imply a negative wealth effect on households that increase their labour supply. The net effect on wages and consumption depends on whether labour demand or labour supply increases more. With nominal rigidities, firms cannot adjust prices but have to satisfy higher demand, so they raise their labour demand by more than in the model without nominal rigidities. As a result, we observe a crowding-in effect on aggregate consumption.

Moreover, the increase in domestic productive government spending induces a rise in the domestic firms' marginal cost and inflation. In particular, we note that the ratio of the price over the marginal cost decreases following the productive spending shock because the marginal cost increases more than inflation, since firms reduce their markup to meet the extra demand at the given prices.

Now we focus on the transmission channels of the productive government spending shock on the trade balance and exchange rate. In Figure 2, the real exchange rate for the domestic country depreciates from the second period onward.[22] Our result is in contrast to the predictions of the

[21] In online Appendix F, we report the IRFs of the model with flexible prices and wages (without nominal rigidities). In online Appendix A, we show the equations for the flexible-price-and-wage version of the model.

[22] In our model, we define the real exchange rate as the price of the foreign consumption basket over the price of the domestic consumption basket in a common currency. Therefore, an upward movement of the real exchange rate in Figure 2 denotes a depreciation for the home country.

Mundell-Fleming model. According to this traditional model, in the presence of a flexible exchange rate regime, a fiscal expansion would lead to an increase in the real interest rate. The higher real interest rate would reduce the net capital outflow. The fall in net capital outflow would reduce the supply of the domestic currency in the foreign exchange market. As a consequence, the domestic exchange rate would appreciate. However, this static interpretation misses the dynamic interaction between fiscal and monetary policy. In response to the shock, the central bank is not aggressive and, as a consequence, the nominal interest rate turns to be negative after one year and half. This implies that the domestic real interest rate falls and the net capital outflow increases. The rise in net capital outflow induces a greater supply of the domestic currency in the market for foreign exchange. As a result, the domestic real exchange rate depreciates.

Focusing on the trade balance, Figure 2 shows that domestic net exports increase.[23] This result is in contrast to the predictions of the Mundell- Fleming model, which postulates that fiscal deficits would induce an appreciation of the domestic exchange rate. This, in turn, would crowd out net exports via a static, relative-price effect: consumers would switch away from domestic goods which are more expensive. However, the Mundell- Fleming model misses an important element, i.e., the intertemporal investment decisions. This can be explained as follows: on impact, the decrease in the national saving caused by an expansionary fiscal policy worsens net exports.[24] At the same time, the increase in productive government spending, largely financed by the capital tax, raises the real rental rate. These effects make investment projects more costly and induce a decrease in the capital stock. In particular, we observe the negative response of the Tobin's q. This is interpreted as the capital shadow price. Its fall indicates that capital is less valuable in the future, so it discourages current investment. Therefore, the increase in productive government spending causes a crowding-out effect on private investment that is more than enough to offset the fall in public savings. Moreover, we observe that national savings turn positive six months after the shock. As a result, the trade balance increases.

Unproductive government expenditure. Figure 3 shows that domestic unproductive public spending exhibits higher persistence compared to domestic productive public spending. This causes a different reaction in the fiscal rules with domestic labour and capital taxes that remain high for a

[23]We note that the response of domestic trade balance is positive from the first quarter onwards.

[24]As in Corsetti and Müller (2006), we refer to the trade balance (or net exports), instead of the current account, as a measure of a country's external position. In this regard, the early literature (e.g., Baxter, 1995) argues that, at business cycle frequencies, the two measures tend to move closely together.

longer period. In particular, the extended increase in the capital tax has a larger effect on private investment and, in turn, this affects net exports more significantly. Despite the increase in distortive taxes and the fall in lump-sum transfers, the higher persistence of this shock induces a much higher response of domestic debt compared to the case of productive spending.

Following the domestic unproductive government spending shock, the reactions of domestic output and hours worked are again positive. On the other hand, the consumption of both domestic and imported goods falls. In fact, a more aggressive monetary policy implies a higher real interest rate and thereby strengthens households' incentives to postpone consumption. Therefore, we observe the crowding-out effect on aggregate consumption.

Focusing on the external sector, the increase in unproductive spending raises the relative price of consumption goods for the domestic country compared to the consumption goods of the foreign country. As noted above, the central bank responds aggressively to this shock by consistently raising the nominal interest rate. As a result, the real interest rate increases enough to induce a net capital inflow. In turn, this implies a lower supply of the domestic currency in the foreign exchange market and a appreciation of the domestic real exchange rate.[25]

The response of the domestic trade balance is systematically positive. Although the stronger persistence of the unproductive spending shock induces a consistent fall in national savings, the worsening in public savings more than offsets the fall in private investment. More specifically, both the increase in the real rental rate and the larger capital tax translate into more expensive investment projects. In this regard, the fall in the Tobin's q, or the capital shadow price, indicates that capital is less valuable in the future. As a consequence, both private capital and investment fall. Since the unproductive government spending shock is more persistent than the productive spending shock, the crowding-out effect on investment lasts a longer period of time. As we mentioned above, this implies that the increase in the trade balance is stronger in this case.

In conclusion, our results show that a positive shock to unproductive government spending induces an appreciation in the domestic currency. This is in line with the traditional analysis based on the Mundell-Fleming model. In contrast with the results of the Mundell-Fleming, we find an improvement in the trade balance of the domestic country. This is because traditional prediction misses an important element by disregarding intertemporal investment decisions. With an unproductive government spending shock, firms expect that capital is less valuable in the future.

[25] As above, given the definition of the real exchange rate in our model, a downward movement of the real exchange rate in Figure 3 denotes an appreciation for the home country.

This crowding-out effect on private investment is more than enough to offset the worsening in the public savings and induces an increase in net exports.

5 Understanding the transmission mechanisms of government spending shocks

In this Section, we analyse the main transmission mechanisms of government spending shocks with specific interest on the real exchange rate and the trade balance. To do so, firstly, we focus on the role of the central bank and, secondly, we look at the role of fiscal policy.

5.1 The role of the central bank

In the previous section, we showed that the response of the real exchange rate to an increase in government spending depends on the type of shock. In particular, we have seen that a productive government spending shock induces a depreciation of the domestic currency. As we already mentioned, this finding is in contrast with the explanations based on the Mundell-Fleming model. According to this traditional theory, an increase in government purchases would lead to an increase in the level of income and, in turn, to an increase in the interest rate. The higher interest rate would reduce net capital outflow and decrease the supply of domestic currency in the market for foreign exchange. Therefore, the domestic exchange rate would appreciate.

Here, we provide a counterfactual analysis to show that in response to an increase in productive government expenditure, our model does not necessarily predict a depreciation of the domestic real exchange rate, but it can even imply an appreciation of the domestic currency. Although we are able to reconcile our results with the explanations of the Mundell-Fleming model, the transmission channels of government spending shocks in our model are different from those implied by the traditional theory. This is because the Mundell-Fleming model misses an important transmission channel of government spending shocks that stems from the dynamic interaction between fiscal and monetary policy.

More specifically, we show that the central bank response to an increase in both types of government expenditure is a key factor for the transmission process of these shocks. Indeed, different degrees of monetary policy activism affect the real interest rate and, in turn, determine the net capital inflow/outflow. In particular, a more aggressive monetary policy implies a higher real interest rate and thereby induces a net capital inflow. Accordingly, the supply of domestic currency reduces, and the real exchange rate appreciates. The Taylor rule, equation (22), plays a central role in determining the response of the nominal interest rate to an increase in inflation caused by expansionary fiscal policy.

Figures 4-5 display the IRFs of some key macroeconomic aggregates in response to productive and unproductive government spending shocks for the benchmark model (black lines) and for a counterfactual model in which the degree of monetary policy activism is higher (red lines).[26] For the latter case, we assume that the Taylor rule coefficient on inflation (r_1^{π}) is three times higher than its estimated value. Our results show that a more aggressive monetary policy implies a positive response of the nominal interest rate for all periods. This, in turn, induces a consistent increase in the real interest rate, and consequently the domestic currency appreciates over time.

5.2 The role of fiscal policy

In Section 4.4, we have seen that the Canadian trade balance improves in response to both productive and unproductive government spending shocks. This finding is in contrast to the Mundell-Fleming model. This traditional model postulates that in response to a positive public spending shock, the appreciation of the domestic exchange rate crowds out net exports due to more expensive domestic goods. However, the Mundell-Fleming model completely ignores the intertemporal investment decisions of the economic agents. Indeed, our model shows that increases in both productive and unproductive government expenditures induce a large crowding-out effect on aggregate investment. This drop in investment more than offsets the deterioration of national savings, such that net exports increase.

In this Section, we provide a counterfactual analysis in which we show that our model is able to reproduce a deterioration in the domestic trade balance after positive public spending shock. However, as highlighted above, the transmission mechanisms of our model are different from those of the Mundell-Fleming. We focus on three main elements that, in our model, generate the crowding-out effect on aggregate investment. The negative effect of an increase in government spending on aggregate investment is due to: (*i*) distortive taxes, (*ii*) quick adjustment of government transfers and (*iii*) the persistence of government spending shocks. Firstly, the presence of distortive taxes in our model implies a negative investment response following the public spending expansion. Our result is not surprising and is in line with Baxter and King (1993), Braun (1994), Leeper and Yang (2008) and Traum and Yang (2015). Secondly, the estimated fiscal rule (16) implies that government

[26] As above, we simulate positive shocks of 1 percent to both productive and unproductive government expenditures.

transfers are substantially reduced in response to a government spending shock. Since Canada has one of the lowest shares of government transfers over GDP among G7 countries[27], a decrease in this fiscal variable reduces economic activity and, thereby, induces a drop in private investment. Finally, in relation to the previous point, a higher persistence of the government spending shock implies a negative response of government transfers for an extended period of time. This, in turn, exacerbates the fall in private investment.

Figures 6-7 show the IRFs of some key macroeconomic fundamentals in response to productive and unproductive government spending shocks for the benchmark model (black lines) and a counterfactual model in which there are neither distortive taxes nor fiscal rules for government transfers and we assume the same low persistence of productive and unproductive spending shocks.[28] Our results show that these three ingredients generate a crowding-in effect of private investment in response to a public spending shock. Since national savings decrease after the shock, the positive response of private investment implies a consistent deterioration of the Canadian trade balance over time.

6 Government spending multipliers and the role of trade elasticity

In this section, we analyse whether the degree of trade elasticity affects the main results described in Section 4.4.

Several recent studies have assessed the role of trade openness in the domestic transmission of government spending. Most of these works have focused on structural panel vector autoregressions (see, for example, Auerbach and Gorodnichenko, 2013 and Faccini et al., 2016). In general, the results found in this literature have been mixed. For example, Ilzetzki et al. (2013), find that public spending multipliers are smaller in countries that are relatively open (i.e., the trade-to-GDP ratio exceeds 60%). On the other hand, Cacciatore and Traum (2022), in a simple two-country, two-good model, show analytically that fiscal multipliers can be larger in economies more open to trade. In line with these studies, our analysis is based on government spending multipliers for domestic output. In particular, we construct present-value multipliers for both types of government spending.

[27] See OECD data at `https://www.oecd.org/en/data/indicators/social-spending.html`.

[28] As above, we simulate positive shocks of 1 percent to both productive and unproductive government expenditures.

Following Leeper et al. (2010b), we have that:

$$Mutliplier = \frac{\sum_{i=0}^{k} \left(\prod_{j=0}^{i} r_{1,t+j}^{-1} \right) \Delta Y_{1,t+i}^{d}}{\sum_{i=0}^{k} \left(\prod_{j=0}^{i} r_{1,t+j}^{-1} \right) \Delta GL_{1,t+i}^{d}}, \tag{27}$$

where $Y_{1,t+i}^{d}$ denotes domestic output and $GL_{1,t+i}^{d}$ can represent either productive ($GP_{1,t+i}^{d}$) or unproductive government spending ($GU_{1,t+i}^{d}$). In equation (27), $\Delta Y_{1,t+i}^{d}$ and $\Delta GL_{1,t+i}^{d}$ indicate changes in the relative level of the variables with respect to their steady-state values. Finally, the discount factor (r_1) represents the real interest rate for the domestic economy.

The top panel of Table 5 shows the cumulative output present-value multipliers based on the mean estimates of our benchmark model, the parameter k determines the period in quarters. We present the results on the impact of the exogenous shock, together with the results for 1, 3, 5 and 10 years ahead and for the infinite-horizon case ($k = 1,000$). In addition, we compute the minimum and maximum values of the respective multipliers. We find present-value multipliers for the output that are within the range of estimated values by Owyang et al. (2013) for the Canadian economy. Notably, our results show that output present-value multipliers are much higher in response to productive government spending shocks than unproductive government expenditure shocks. This is especially true in the long run. In line with Angelini et al. (2023), one year after the productive government spending shock occurred, the multiplier exceeds one.[29]

As a counterfactual exercise, we try alternative values for the Canadian trade elasticity, $\frac{1+\rho_1^c}{\rho_1^c} = \frac{1+\rho_1^i}{\rho_1^i}$, and check whether and how the output multipliers are affected by these changes. As we mentioned in Section 4, our benchmark value aligns with estimates reported in the literature. In our experiment, we consider two additional cases: (i) a low degree of trade openness corresponding to a value of the Canadian trade elasticity equal to 0.40; (ii) a high degree of trade openness corresponding to a value of the Canadian trade elasticity equal to 1.60. In contrast to the papers cited above, we disaggregate public spending into productive and unproductive expenditures. Accordingly, we can evaluate the cumulative present-value multipliers for output in both cases.

We start by focusing on the productive government spending shock. As we can see in Table 5, in the short run, the present-value multipliers for output have the highest values in the case of the model with a low degree of trade elasticity. On the contrary, in the long run, the output multipliers

[29] Note that in Angelini et al. (2023), government spending multipliers are estimated for the US economy.

are larger for both the benchmark model and the model with a high degree of trade openness. This result is explained by the responses of both domestic consumption and domestic investment.[30] In the short run, with a low degree of trade openness, domestic consumption increases more, and domestic investment decreases less with respect to the benchmark model and the model with a high degree of trade elasticity. Instead, in the long run, a more open economy has a larger increase in domestic consumption and a lower drop in domestic investment.

Turning to the unproductive spending shock, Table 5 shows that the present-value multipliers for output are consistently larger in the case of low trade elasticity. Again, this is due to the responses of both domestic consumption and domestic investment. In particular, the model with a low degree of trade openness implies a lower fall in both these macroeconomic aggregates compared to the benchmark model and the model with a high degree of trade elasticity.[31] Therefore, our results indicate that the degree of trade openness affects public spending multipliers according to the type of government expenditure shock. In the case of an unproductive public spending shock, a more open economy presents smaller output multipliers. In the case of a productive spending shock, the output multipliers are larger in a more open economy only in the long run.

7 Conclusions

In this paper, we propose an open economy general equilibrium model that includes international trade between Canada and the US. We estimate our model using the Mixture of Student's t by Importance Sampling Weighted Expectation Maximization (MitISEM). This algorithm applies IS to compute the unknown posterior density by a mixture of Student's t densities and adapts the parameters to the most recent IS draws via an EM step. This algorithm can be easily parallelized and does not require tuning parameters for the user. Moreover, it is flexible and can handle different unknown and complex forms in a reasonable computing time.

Our findings show that an increase in productive spending generates a consistent increase in domestic output and hours worked, as well as consumption increases. In response to this shock, the domestic real exchange rate depreciates due to the weak central bank response to an increase in domestic inflation. We also observe an increase in domestic net exports because the crowding-out effect on private investment is larger than the fall in public savings. A positive shock to

[30] Online Appendix G shows the present-value multipliers for both domestic consumption and domestic investment in the case of a productive government spending shock.

[31] In online Appendix G, we report the present-value multipliers for both domestic consumption and domestic investment in the case of an unproductive public spending shock.

unproductive spending induces a positive response of domestic output and hours worked. However, in this case, the negative wealth effect induces a fall in aggregate consumption. As in the case of a productive spending shock, the trade balance improves. However, the domestic real exchange rate appreciates in line with the predictions of the Mundell-Fleming model.

We analyse present-value multipliers for domestic output. Such multipliers are larger in response to productive government spending shocks than unproductive government expenditure shocks. We also show that the degree of trade openness matters in terms of transmission channels of productive and productive government spending shocks on the economy. In the short run, the present-value multipliers for output have the highest values in the case of the model with a low degree of trade elasticity. On the contrary, in the long run, the output multipliers are larger for both the benchmark model and the model with a high degree of trade openness. This result suggests that while policies aimed at restricting international trade may yield short-term benefits, they can be harmful in the long run.

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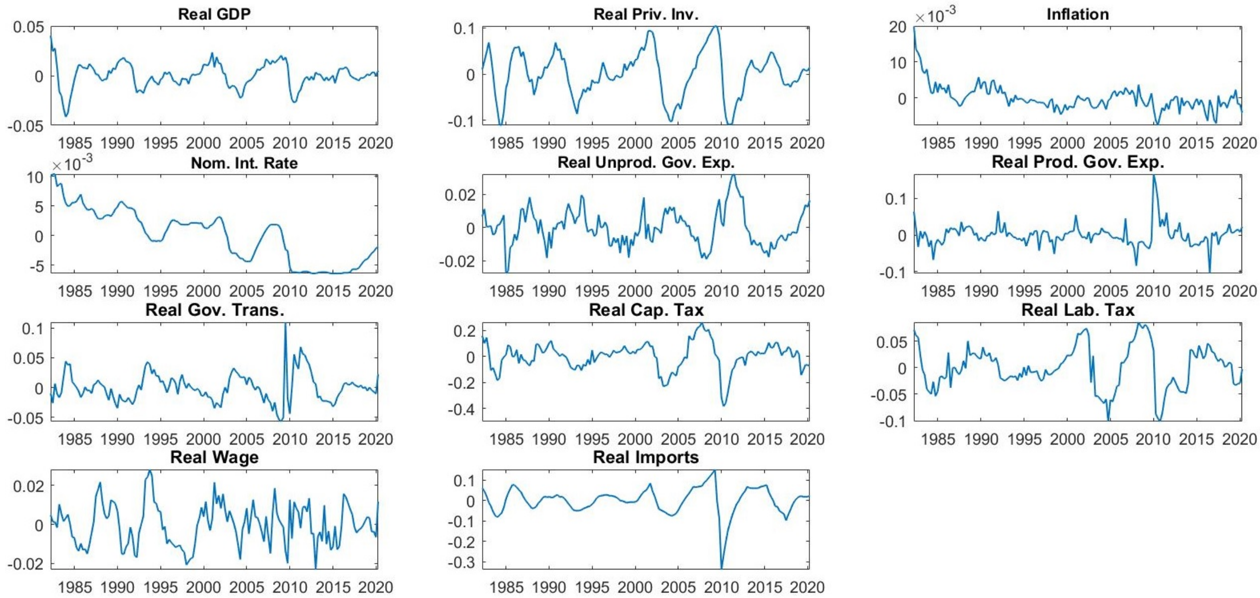
Canada



US

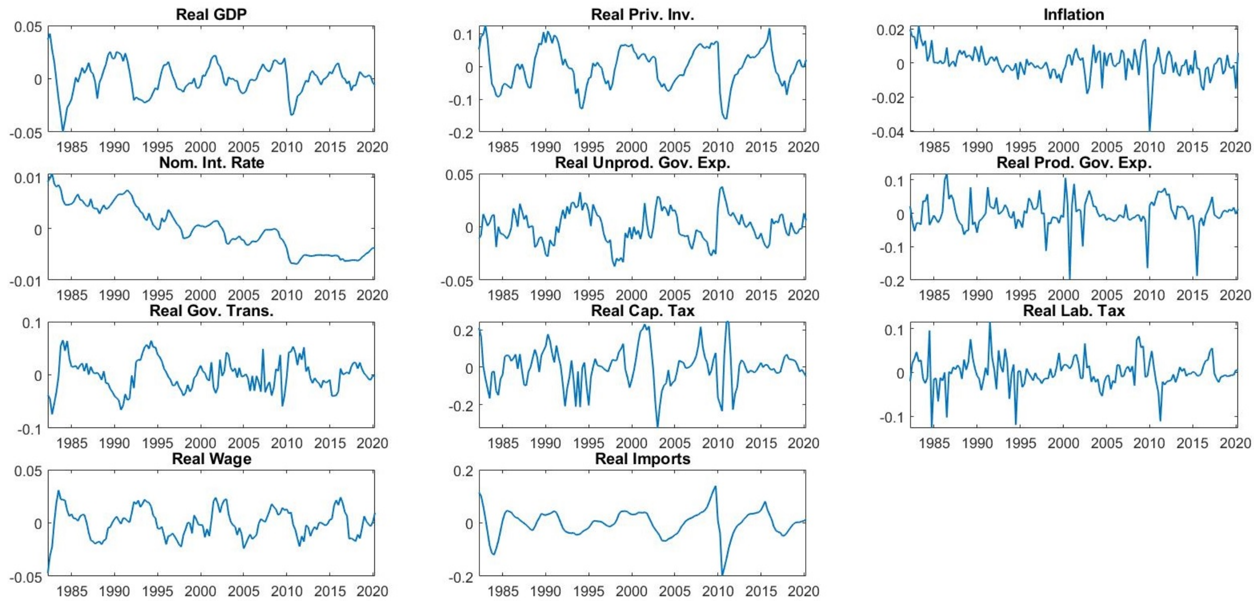


Figure 1: *Transformed data used in the estimation*

Notes: In the graphs above, the blue lines indicate the observed data used to estimate our model. The sample is 1981:Q1-2019:Q1.

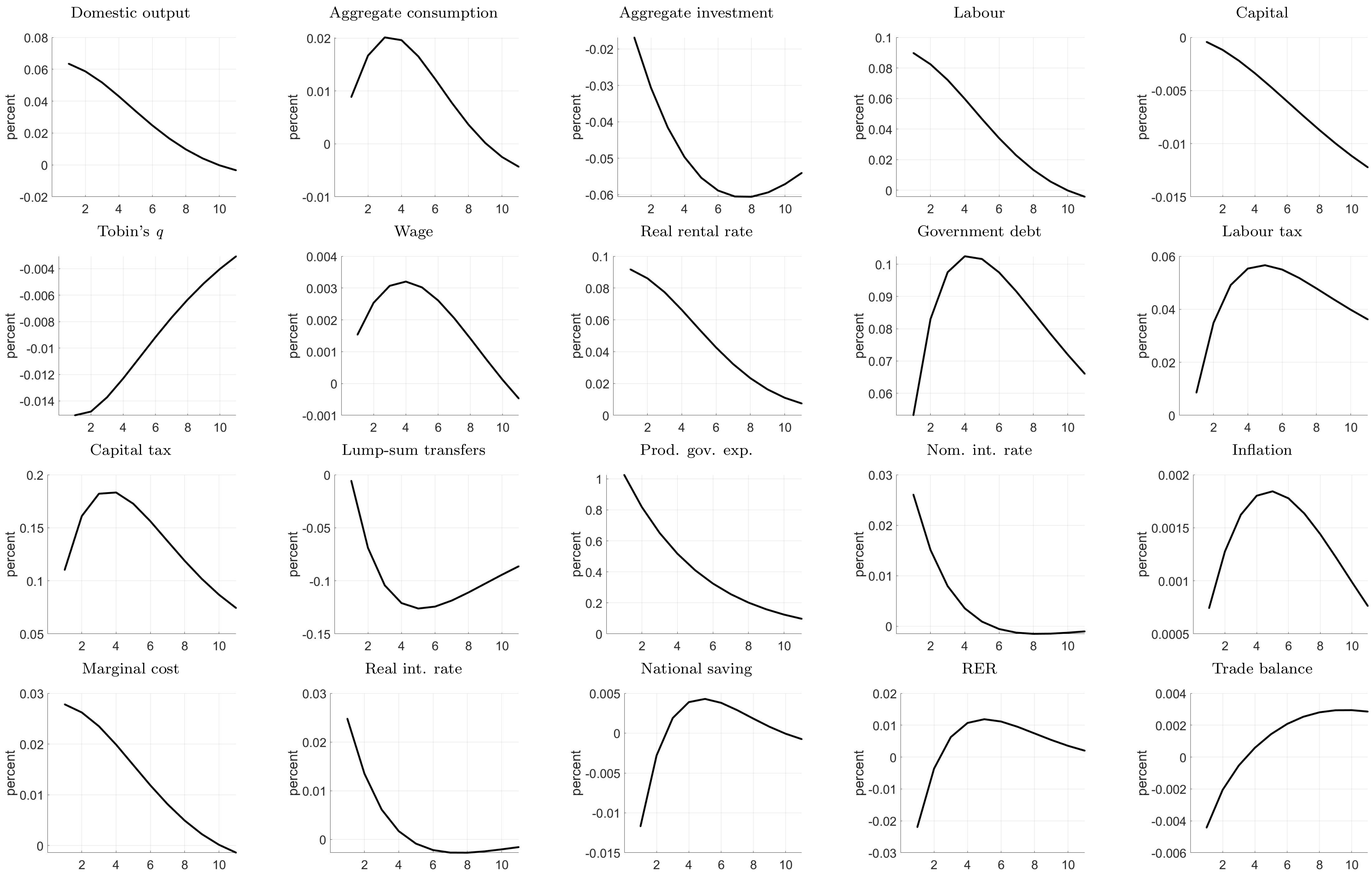


Figure 2: *Impulse responses to a productive government spending shock*

Notes: One percent increase in domestic productive public spending for the Canadian economy. X-axis is in quarters.

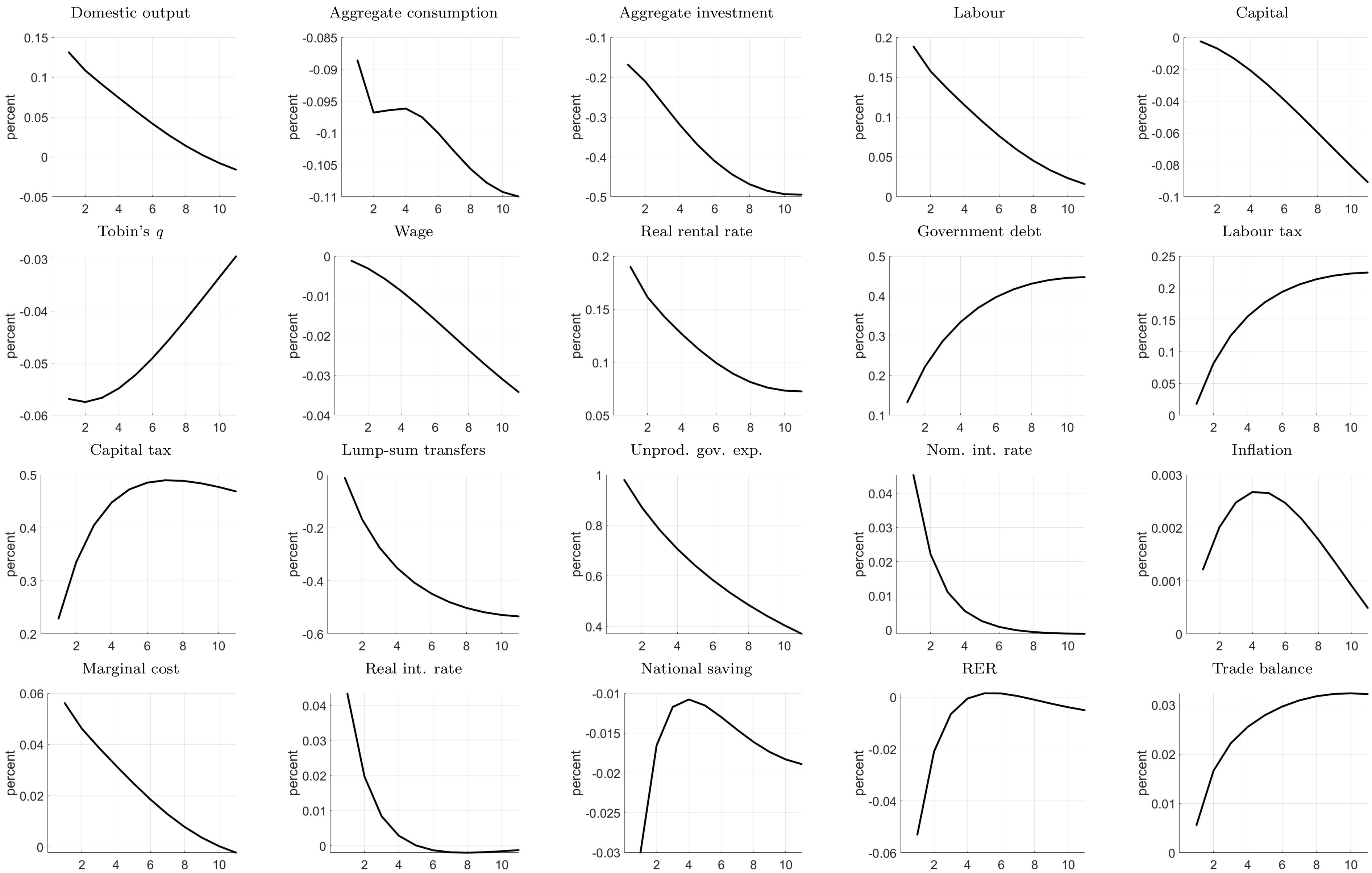


Figure 3: *Impulse responses to an unproductive government spending shock*

Notes: One percent increase in domestic unproductive public spending for the Canadian economy. X-axis is in quarters.

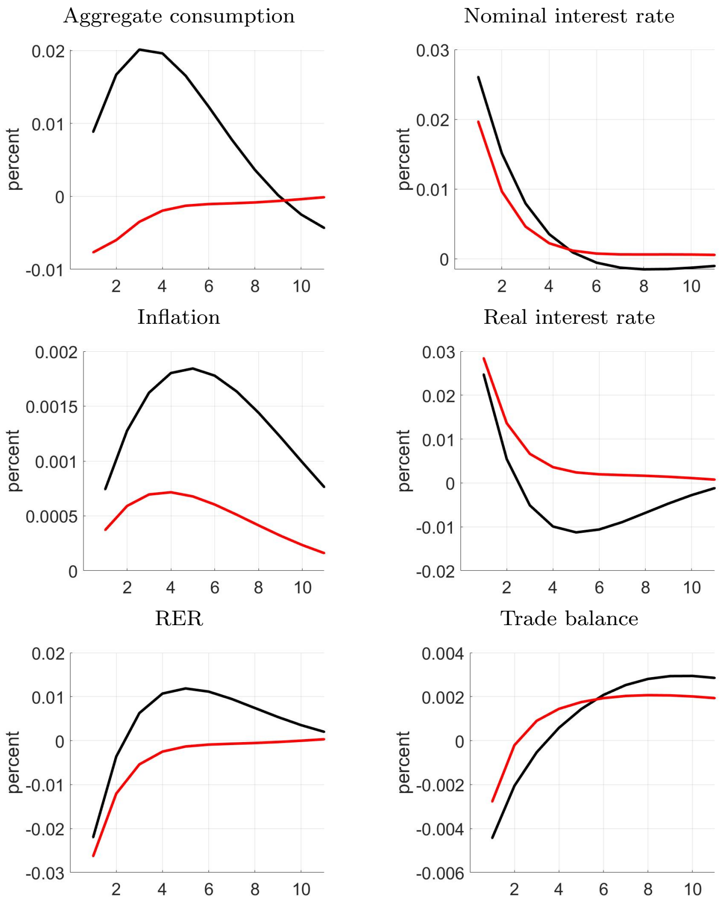


Figure 4: *The role of monetary policy - IRFs to a 1% increase in domestic productive public spending for the Canadian economy*

Notes: Solid black lines: benchmark model (estimated value for r_1^{π}); Solid red lines: aggressive monetary policy ($r_1^{\pi} \times 3$). X-axis is in quarters.

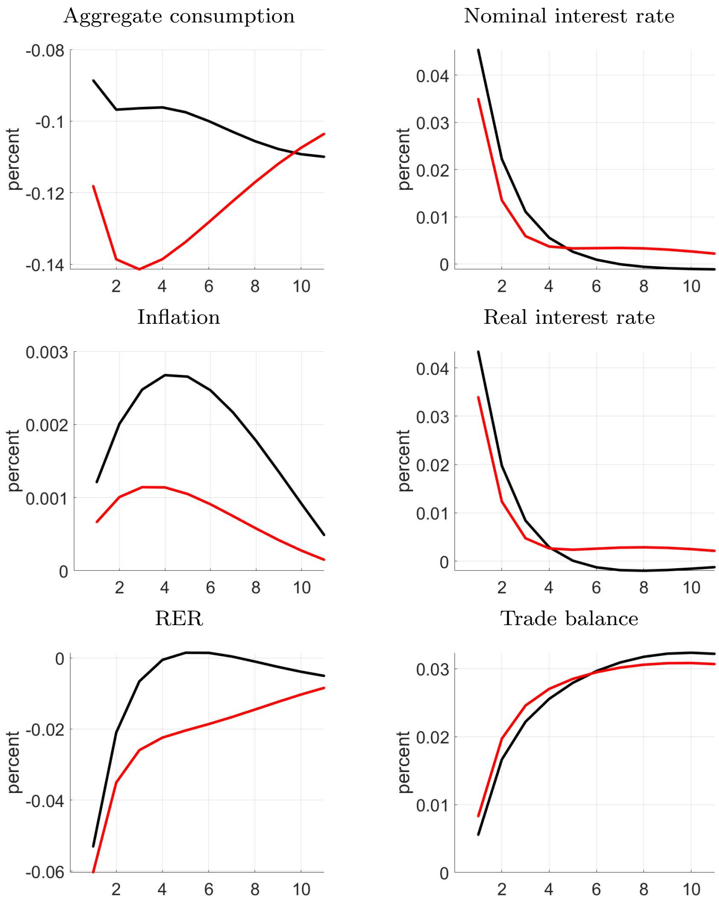


Figure 5: *The role of monetary policy - IRFs to a 1% increase in domestic unproductive public spending for the Canadian economy*

Notes: Solid black lines: benchmark model (estimated value for r_1^{π}); Solid red lines: aggressive monetary policy ($r_1^{\pi} \times 3$). X-axis is in quarters.

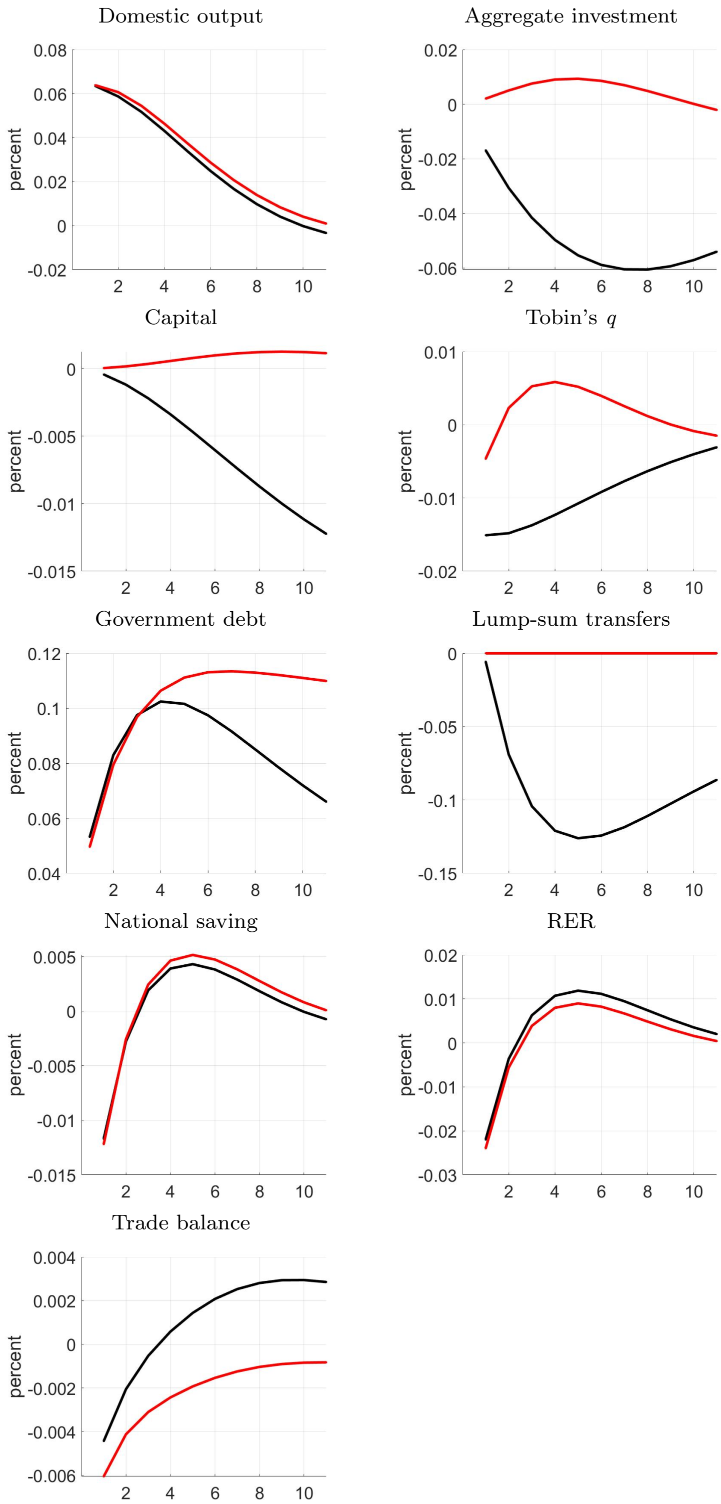


Figure 6: *The role of fiscal policy - IRFs to a 1% increase in domestic productive public spending for the Canadian economy*

Notes: Solid black lines: benchmark model; Solid red lines: model with no distortive taxes, no fiscal rules for government transfers ($\phi_1^{yt} = 0$ and $\gamma_1^{bt} = 0$) and same persistence of productive and unproductive spending shocks ($\rho_1^{gp} = \rho_1^{gu} = 0.79$). X-axis is in quarters.

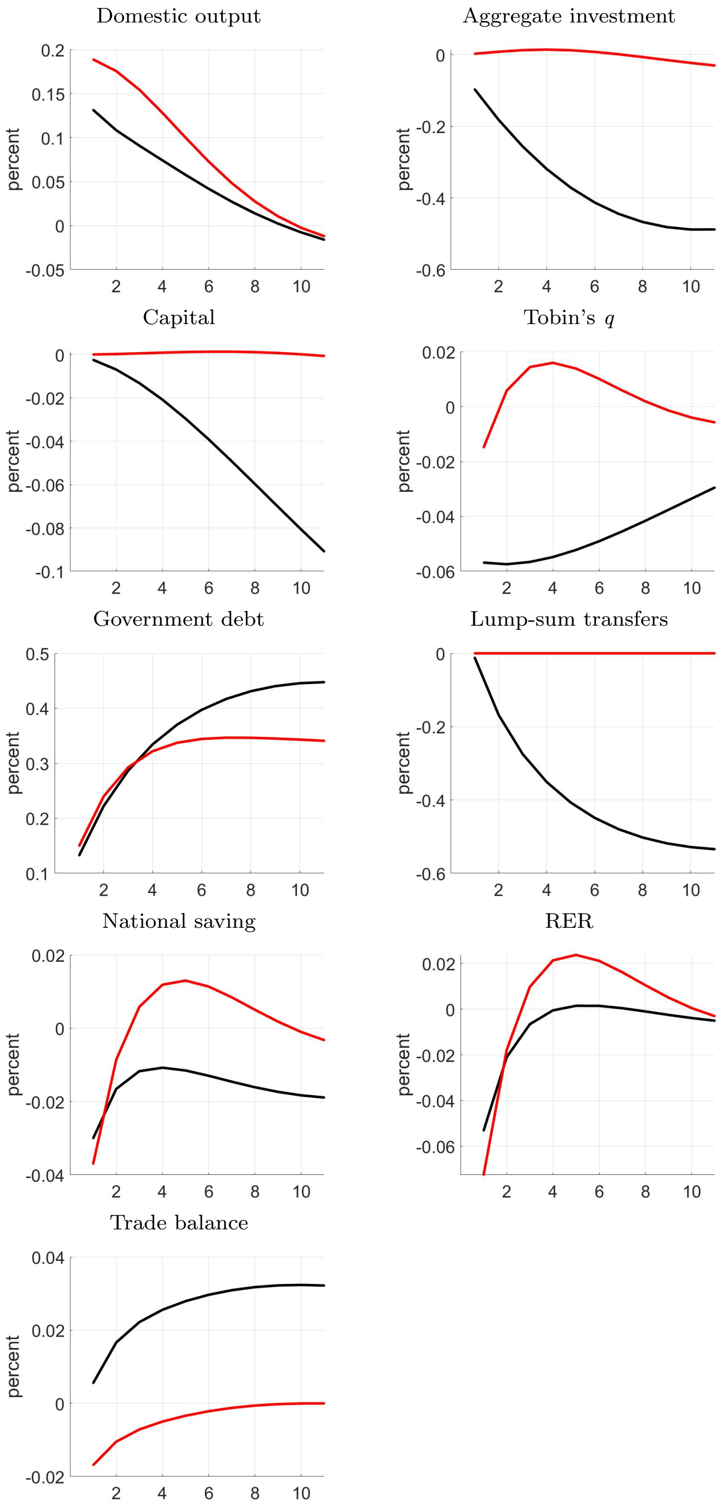


Figure 7: *The role of fiscal policy - IRFs to a 1% increase in domestic unproductive public spending for the Canadian economy*

Notes: Solid black lines: benchmark model; Solid red lines: model with no distortive taxes, no fiscal rules for government transfers ($\phi_1^{yt} = 0$ and $\gamma_1^{bt} = 0$) and same persistence of productive and unproductive spending shocks ($\rho_1^{gp} = \rho_1^{gu} = 0.79$). X-axis is in quarters.

Table 1: *Calibrated parameters according to the literature for the open-economy model*

Full Name	Symbol	Value	Source
Discount Factor	β_i	0.9960	Del Negro and Schorfheide (2008)
Depreciation Rate of Priv. Cap.	δ_i	0.0250	Ann. Cap. Depr: 0.10
Intertemp. Elas. of Sub.	$\frac{1}{\sigma_i^c}$	0.2000	Jermann (1998)
Elast. Labour Supply	σ_i^l	4.0000	Chetty et al. (2013)
S.S. Mark-up in Goods Market	ν_i^p	1.5000	Smets and Wouters (2007)
S.S. Mark-up in Lab. Market	ν_i^w	1.5000	Smets and Wouters (2007)
Goods Market Agg. Cur.	ϑ_i^p	10.0000	Smets and Wouters (2007)
Lab. Market Agg. Cur.	ϑ_i^w	10.0000	Smets and Wouters (2007)
Bond Intermediation Cost	ϕ_i^b	0.0001	Bodenstein et al. (2011)
Depreciation Rate of Gov. Cap.	δ_i^g	0.0050	Leeper et al. (2010b)
Priv. Cap. Share in Prod.	α_i^k	0.3000	Leeper et al. (2010b)
Pub. Cap. Share in Prod.	α_i^{kg}	0.1500	Asimakopoulos et al. (2021)

Notes: The table shows the parameter's name (Full name), the acronym symbol (Symbol), the calibrated value (Value) and the source of the parameter (Source). The subscript $i = \{Canada, US\}$ indicates that the parameters have the same values for both countries.

Table 2: *Calibrated parameters according to observed data of Canada and the US*

Parameter Description	**Canada**	**US**
Unprod. Gov. Exp. / GPD	$\frac{\left(GU_1^d\right)^{SS}}{\left(Y_1^d\right)^{SS}} = 0.210$	$\frac{\left(GU_2^d\right)^{SS}}{\left(Y_2^d\right)^{SS}} = 0.153$
Prod. Gov. Exp. / GPD	$\frac{\left(GP_1^d\right)^{SS}}{\left(Y_1^d\right)^{SS}} = 0.069$	$\frac{\left(GP_2^d\right)^{SS}}{\left(Y_2^d\right)^{SS}} = 0.073$
Gov. Transfers / GDP	$\frac{(T_1)^{SS}}{\left(Y_1^d\right)^{SS}} = 0.101$	$\frac{(T_2)^{SS}}{\left(Y_2^d\right)^{SS}} = 0.117$
S.S. Capital Tax Rate	$\left(\tau_1^k\right)^{SS} = 0.341$	$\left(\tau_2^k\right)^{SS} = 0.228$
S.S. Labour Tax Rate	$\left(\tau_1^l\right)^{SS} = 0.360$	$\left(\tau_1^l\right)^{SS} = 0.259$
Weight of Cons. in Tot. Imp.	$\omega_1^{mc} = 0.223$	$\omega_2^{mc} = 0.025$
Weight of Services in Tot. Imp.	$\omega_1^{mi} = 0.944$	$\omega_2^{mi} = 0.095$
Population Size	$\zeta_1 = 0.099$	$\zeta_2 = 0.901$

Notes: The table shows the calibrated parameters according to actual data. We report the parameter values (Parameter) for Canada and the US (see online Appendix A for a detailed description of the construction of several parameters).

Table 3: *Priors and posteriors for the endogenous parameters of the open-economy model*

				Canada		**US**	
Full Name	Symbol	Prior	(Mean, St. Dev.)	Mean	[5%, 95%]	Mean	[5%, 95%]
Cons. Habit Pers.	h_i	$\mathcal{B}$	(0.70, 0.05)	0.38	[0.36, 0.41]	0.20	[0.17, 0.22]
Inv. Adjustment Cost	φ_i^{i}	$\mathcal{G}$	(4.00, 1.50)	4.93	[4.91, 4.95]	7.52	[7.49, 7.55]
Calvo Wages Prob.	ξ_i^w	$\mathcal{B}$	(0.60, 0.20)	0.90	[0.88, 0.92]	0.87	[0.84, 0.90]
Calvo Prices Prob.	ξ_i^p	$\mathcal{B}$	(0.60, 0.20)	0.82	[0.80, 0.84]	0.87	[0.84, 0.89]
Degree of Wage Ind.	ι_i^w	$\mathcal{B}$	(0.50, 0.15)	0.96	[0.94, 0.98]	0.63	[0.60, 0.65]
Degree of Price Ind.	ι_i^p	$\mathcal{B}$	(0.50, 0.15)	0.97	[0.94, 0.99]	0.18	[0.16, 0.20]
Int. Rate Smoothing	ρ_i	$\mathcal{B}$	(0.50, 0.20)	0.41	[0.39, 0.44]	0.61	[0.58, 0.63]
T.R. Coef. on Inf.	r_i^π	$\mathcal{G}$	(2.00, 0.25)	1.81	[1.78, 1.83]	1.10	[1.08, 1.12]
T.R. L.R. Coef. on Y	r_i^y	$\mathcal{G}$	(0.20, 0.10)	0.05	[0.03, 0.07]	0.03	[0.01, 0.05]
T.R. S.R. Coef. on Y	$r_i^{\Delta_y}$	$\mathcal{G}$	(1.20, 0.05)	0.47	[0.45, 0.49]	0.50	[0.48, 0.52]
τ^l/Y Coef.	ϕ_i^{yl}	$\mathcal{G}$	(0.10, 0.05)	0.13	[0.11, 0.16]	0.20	[0.17, 0.23]
τ^k/Y Coef.	ϕ_i^{yk}	$\mathcal{G}$	(0.40, 0.20)	1.74	[1.70, 1.76]	2.62	[2.59, 2.65]
T/Y Coef.	ϕ_i^{yt}	$\mathcal{G}$	(0.10, 0.05)	0.09	[0.06, 0.11]	0.08	[0.06, 0.10]
τ^l/B Coef.	γ_i^{bl}	$\mathcal{G}$	(0.05, 0.04)	0.51	[0.48, 0.54]	0.39	[0.37, 0.41]
τ^k/B Coef.	γ_i^{bk}	$\mathcal{G}$	(0.30, 0.15)	1.11	[1.09, 1.14]	0.50	[0.46, 0.53]
T/B Coef.	γ_i^{bt}	$\mathcal{G}$	(0.50, 0.20)	1.20	[1.17, 1.23]	3.85	[3.83, 3.87]
G^p/YCoef.	ϕ_i^{ygp}	$\mathcal{G}$	(0.15, 0.05)	0.42	[0.39, 0.45]	0.51	[0.49, 0.53]
G^u/Y Coef.	ϕ_i^{ygu}	$\mathcal{G}$	(0.15, 0.05)	0.15	[0.13, 0.17]	3.52	[3.50, 3.54]
G^p/B Coef.	γ_i^{bgp}	$\mathcal{G}$	(0.40, 0.20)	0.02	[0.01, 0.04]	0.29	[0.29, 0.32]
G^u/B Coef.	γ_i^{bgu}	$\mathcal{G}$	(0.40, 0.20)	0.43	[0.41, 0.45]	0.02	[0.01, 0.03]
Cons. / Inv. Import Sub. El.	$\frac{1+\rho_i^c}{\rho_i^c} = \frac{1+\rho_i^i}{\rho_i^i}$	$\mathcal{N}$	(1.10, 0.10)	1.07	[1.07, 1.07]	1.01	[1.01, 1.01]

Notes: The table shows the posterior means and credible intervals for the $5th$ and $95th$ percentiles. We also report the prior means and standard deviations of the endogenous parameters. Regarding the prior distributions of the endogenous parameters, $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{G}$ stand for Beta, Normal and Gamma, respectively. The subscript $i = \{Canada, US\}$ indicates that the parameters have the same priors for both countries.

Table 4: *Priors and posteriors for the exogenous parameters of the open-economy model*

				Canada		**US**	
Full Name	Symbol	Prior	(Mean, St. Dev.)	Mean	[5%, 95%]	Mean	[5%, 95%]
Investment Pers.	ρ_i^i	$\mathcal{B}$	(0.75, 0.15)	0.43	[0.40, 0.46]	0.69	[0.66, 0.72]
Imp. Pref. Pers.	ρ_i^m	$\mathcal{B}$	(0.75, 0.15)	0.94	[0.92, 0.96]	0.85	[0.82, 0.87]
Wage Mark-up Pers.	ρ_i^w	$\mathcal{B}$	(0.75, 0.15)	0.52	[0.50, 0.55]	0.79	[0.78, 0.82]
Price Mark-up Pers.	ρ_i^p	$\mathcal{B}$	(0.75, 0.15)	0.52	[0.50, 0.55]	0.15	[0.13, 0.17]
Productivity Pers.	ρ_i^a	$\mathcal{B}$	(0.75, 0.15)	0.89	[0.87, 0.92]	0.39	[0.36, 0.41]
Prod. Gov. Exp. Pers.	ρ_i^{gp}	$\mathcal{B}$	(0.75, 0.15)	0.79	[0.77, 0.82]	0.79	[0.77, 0.81]
Unprod. Gov. Exp. Pers.	ρ_i^{gu}	$\mathcal{B}$	(0.75, 0.15)	0.94	[0.92, 0.97]	0.91	[0.89, 0.94]
Gov. Transfers Pers.	ρ_i^t	$\mathcal{B}$	(0.75, 0.15)	0.70	[0.67, 0.72]	0.92	[0.89, 0.94]
Capital Tax Pers.	ρ_i^k	$\mathcal{B}$	(0.75, 0.15)	0.75	[0.73, 0.77]	0.91	[0.88, 0.94]
Labour Income Tax Pers.	ρ_i^l	$\mathcal{B}$	(0.75, 0.15)	0.54	[0.52, 0.57]	0.87	[0.85, 0.89]
Monetary Policy Pers.	ρ_i^r	$\mathcal{B}$	(0.75, 0.15)	0.14	[0.12, 0.16]	0.21	[0.19, 0.23]
Investment St. Err.	σ_i^i	$\mathcal{IG}$	(0.10, 2.00)	0.06	[0.04, 0.08]	0.17	[0.14, 0.20]
Imp. Pref. St. Err.	σ_i^m	$\mathcal{IG}$	(0.10, 2.00)	0.09	[0.07, 0.11]	1.08	[1.05, 1.10]
Wage Mark-up St. Err.	σ_i^w	$\mathcal{IG}$	(0.10, 2.00)	0.02	[0.01, 0.04]	0.14	[0.12, 0.16]
Price Mark-up St. Err.	σ_i^p	$\mathcal{IG}$	(0.10, 2.00)	0.03	[0.02, 0.05]	0.04	[0.03, 0.06]
Productivity St. Err.	σ_i^a	$\mathcal{IG}$	(0.10, 2.00)	0.19	[0.16, 0.21]	0.58	[0.56, 0.61]
Prod. Gov. Exp. St. Err.	σ_i^{gp}	$\mathcal{IG}$	(0.10, 2.00)	0.05	[0.04, 0.08]	0.03	[0.02, 0.04]
Unprod. Gov. Exp. St. Err.	σ_i^{gu}	$\mathcal{IG}$	(0.10, 2.00)	0.05	[0.03, 0.08]	0.03	[0.02, 0.04]
Gov. Transfers St. Err.	σ_i^t	$\mathcal{IG}$	(0.10, 2.00)	0.02	[0.01, 0.04]	0.05	[0.03, 0.07]
Capital Tax St. Err.	σ_i^k	$\mathcal{IG}$	(0.10, 2.00)	0.06	[0.04, 0.08]	0.04	[0.02, 0.05]
Labour Income Tax St. Err.	σ_i^l	$\mathcal{IG}$	(0.10, 2.00)	0.03	[0.02, 0.05]	0.02	[0.01, 0.03]
Monetary Policy St. Err.	σ_i^r	$\mathcal{IG}$	(0.10, 2.00)	0.03	[0.02, 0.04]	0.09	[0.07, 0.11]

Notes: The table shows the posterior means and credible intervals for the $5th$ and $95th$ percentiles. We also report the prior means and standard deviations of the endogenous parameters. Regarding the prior distributions of the endogenous parameters, $\mathcal{B}$ and $\mathcal{IG}$ stand for Beta and Inverse Gamma, respectively. The subscript $i = \{Canada, US\}$ indicates that the parameters have the same priors for both countries.

Table 5: *Government spending multipliers and the role of trade elasticity*

Var.	*Impact*	*1-yr*	*3-yrs*	*5-yrs*	*10-yrs*	∞	[min, max]
Benchmark Model							
Productive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GP^d_{1,t+i}}$	0.8928	1.0058	0.9987	0.9365	0.8734	0.8598	[0.8582, 1.0249]
Unproductive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GU^d_{1,t+i}}$	0.6359	0.5852	0.4667	0.3666	0.2254	0.0655	[0.0655, 0.6359]
Model with Low Trade Elasticity							
Productive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GP^d_{1,t+i}}$	0.9516	1.0308	0.9933	0.9265	0.8618	0.8451	[0.8448, 1.0360]
Unproductive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GU^d_{1,t+i}}$	0.6786	0.6086	0.4784	0.3749	0.2328	0.0730	[0.0730, 0.6786]
Model with High Trade Elasticity							
Productive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GP^d_{1,t+i}}$	0.8411	0.9727	0.9887	0.9315	0.8701	0.8590	[0.8411,1.0052]
Unproductive Government Spending Present-Value Multipliers							
$\frac{\Delta Y^d_{1,t+i}}{\Delta GU^d_{1,t+i}}$	0.6016	0.5626	0.4553	0.3568	0.2169	0.0573	[0.0573,0.6016]

Notes: The table shows the cumulative present-value multipliers for output based on the estimated model where $Y^d_{1,t+i}$ denotes domestic output, $GP^d_{1,t+i}$ is the productive government spending, $GU^d_{1,t+i}$ is the unproductive government spending and Δs indicate the relative level changes of the variables with respect to their steady-state values. In the benchmark model, $\frac{1+\rho^c_1}{\rho^c_1} = \frac{1+\rho^i_1}{\rho^i_1}$ is equal to its estimated value. In the model with low trade elasticity, we assume $\frac{1+\rho^c_1}{\rho^c_1} = \frac{1+\rho^i_1}{\rho^i_1} = 0.40$. In the model with high trade elasticity, we assume $\frac{1+\rho^c_1}{\rho^c_1} = \frac{1+\rho^i_1}{\rho^i_1} = 1.60$.

Appendix

This Appendix describes the MitISEM scheme. The main objective of the MitISEM is to provide an automatic and flexible method to construct a candidate density minimizing the Kullback-Leibler divergence between two densities: the target density, and the candidate density. To construct a good candidate, a mixture of Student's t that efficiently cover the target density is estimated. The modes, scales, degrees of freedom and mixing probabilities are quickly optimized using the importance sampling (IS) weighted expectation maximization (EM) method.

Let us define $f(\theta|y)$ as the target density kernel of θ, the k-dimensional vector of interest conditioning on the data. To simplify the notation, we use $f(\theta)$. Let $g(\theta)$ be a candidate density, a mixture of H Student t densities such that:

$$g(\theta) = g\left(\theta|\mu_h, \Sigma_h, \nu_h\right) = \sum_{h=1}^{H} \eta_h t_k\left(\theta|\mu_h, \Sigma_h, \nu_h\right), \tag{28}$$

where μ_h is a location parameter, Σ_h is a scale matrix, and ν_h is the degrees of freedom. Finally, η_h is the mixing probability of the k-dimensional Student's t components with density:

$$t_k(\theta|\mu_h, \Sigma_h, \nu_h) = \frac{\Gamma\left(\frac{\nu_h+k}{2}\right)}{\Gamma\left(\frac{\nu_h}{2}\right)(\pi\nu_h)^{k/2}}|\Sigma_h|^{-1/2}\left(1+\frac{(\theta-\mu_h)^{\top}\Sigma_h^{-1}(\theta-\mu_h)}{\nu_h}\right)^{-(k+\nu_h)/2}, \tag{29}$$

with $h = 1, \ldots, H$ and Σ_h is positive definite, $\eta_h \geq 0$ and $\sum_{h=1}^{H} \eta_h = 1$. The ν_h is restricted to be $\nu_h \geq 0.01$.

The MitISEM relies on the iterative construction of a mixture of Student's t as the candidate density, minimizing the Kullback-Leibler divergence between target and candidate densities. All parameters $(\mu_h, \Sigma_h, \nu_h, \eta_h)$ are jointly optimised using an EM algorithm. This implies a large reduction in computational time and a better candidate in most applications.

The EM algorithm (Dempster et al., 1977) is a method to achieve the maximum likelihood estimates of the parameters θ in models with incomplete data or latent variables. If the latent variables were observable, the computation of the maximum likelihood estimate of θ would be relatively straightforward, depending on the degree of nonlinearity of the first-order conditions. The idea behind EM is to take the expectation of the objective function, in most cases the log-likelihood function, with respect to the latent variables. The expectation of the log-likelihood function is then maximized with respect to the model parameters. In many models, expectations of the latent variables depend on the model parameters θ, hence the two steps are repeated until convergence.

In MitISEM, the EM is used to find the optimal mixture of Student's t densities for a given set of draws from a previous candidate (and their corresponding weights). We apply an IS-weighted EM algorithm to these candidate draws instead of a regular EM algorithm to posterior draws (obtained by applying the Metropolis-Hastings method to these candidate draws), since the former has three advantages. Firstly, we do not require a burn-in sample. Secondly, the use of all candidate draws (without the rejections of the MH method) helps to prevent numerical problems with estimating scale matrices of Student's t components; also, draws with relatively small, yet positive importance weights are helpful for this purpose. Thirdly, the use of all candidate draws may lead to a better approximation.

Following Hoogerheide et al. (2012b), the EM algorithm with IS weights is given by:

$$\tilde{z}_h^i \equiv \mathrm{E}\left[z_h^i \middle| \theta^i, \mu_h^{(l-1)}, \Sigma_h^{(l-1)}, \nu_h^{(l-1)} \right] = \frac{t_k\left(\theta^i | \mu_h, \Sigma_h, \nu_h\right)\ \eta_h}{\sum_{j=1}^H t_k\left(\theta^i | \mu_j, \Sigma_j, \nu_j\right)\ \eta_j}, \tag{30}$$

$$\widetilde{z/w}_h^i \equiv \mathrm{E}\left[\frac{z_h^i}{w_h^i} \middle| \theta^i, \mu_h^{(l-1)}, \Sigma_h^{(l-1)}, \nu_h^{(l-1)} \right] = \tilde{z}_h^i \frac{k+\nu_h}{\rho_h^i + \nu_h}, \tag{31}$$

$$\begin{aligned} \xi_h^i &\equiv \mathrm{E}\left[\log w_h^i \middle| \theta^i, \mu_h^{(l-1)}, \Sigma_h^{(l-1)}, \nu_h^{(l-1)} \right] = \\ &= \left[\log\left(\frac{\rho_h^i + \nu_h}{2} \right) - \psi\left(\frac{k+\nu_h}{2} \right) \right] \tilde{z}_h^i + \left[\log\left(\frac{\nu_h}{2} \right) - \psi\left(\frac{\nu_h}{2} \right) \right] (1 - \tilde{z}_h^i), \end{aligned} \tag{32}$$

$$\delta_h^i \equiv \mathrm{E}\left[\frac{1}{w_h^i} \middle| \theta^i, \mu_h^{(l-1)}, \Sigma_h^{(l-1)}, \nu_h^{(l-1)} \right] = \frac{k+\nu_h}{\rho_h^i + \nu_h}\ \tilde{z}_h^i + (1 - \tilde{z}_h^i), \tag{33}$$

where i are the draws; $\rho_h^i \equiv (\theta^i - \mu_h)^\top \Sigma_h^{-1} (\theta^i - \mu_h)$; $\psi(\cdot)$ is the digamma function (the derivative of the logarithm of the gamma function $\log \Gamma(\cdot)$); $\mu_h^{(l-1)}, \Sigma_h^{(l-1)}, \nu_h^{(l-1)}, \eta_h^{(l-1)}$ are the parameters optimized in the previous $(l-1)$ EM step.

Given the expectation of the latent variables in equation (30) to (33), the parameters of each mixture component are updated using the first order conditions of the expectation of the objective function in the maximization step:

$$\mu_h^{(l)} = \left[\sum_{i=1}^{N} W^i \ \widetilde{z/w}_h^i\right]^{-1} \left[\sum_{i=1}^{N} W^i \ \widetilde{z/w}_h^i \ \theta^i\right], \tag{34}$$

$$\hat{\Sigma}_h^{(l)} = \frac{\sum_{i=1}^{N} W^i \ \widetilde{z/w}_h^i \left(\theta^i - \mu_h^{(l)}\right)\left(\theta^i - \mu_h^{(l)}\right)^\top}{\sum_{i=1}^{N} W^i \ \tilde{z}_h^i}, \tag{35}$$

$$\eta_h^{(l)} = \frac{\sum_{i=1}^{N} W^i \ \tilde{z}_h^i}{\sum_{i=1}^{N} W^i}. \tag{36}$$

where $W^i \equiv f(\theta^i)/g_0(\theta^i)$ are the IS weights.

Finally, $\nu_h^{(l)}$ is solved from the first order condition of ν_h:

$$-\psi(\nu_h/2) + \log(\nu_h/2) + 1 - \frac{\sum_{i=1}^{N} W^i \ \xi_h^i}{\sum_{i=1}^{N} W^i} - \frac{\sum_{i=1}^{N} W^i \ \delta_h^i}{\sum_{i=1}^{N} W^i} = 0. \tag{37}$$

MitISEM optimises the degree-of-freedom parameter ν_h during the EM procedure to obtain a better approximation of the target density. Furthermore, the resulting values of ν_h $(h = 1, \ldots, H)$ can provide information on the shape, for example kurtosis of the target distribution.

MitISEM: Detailed algorithm

The MitISEM approach for obtaining an approximation to a target density:

(1) **Initialization**: Simulate draws $\theta^1, \ldots, \theta^N$ from a "naive" candidate distribution with density g_{naive}, which is obtained as follows. Firstly, we simulate candidate draws from a Student's t distribution with density g_{mode}, where the mode is taken equal to the mode of the target density and the scale matrix equal to minus the inverse Hessian of the log-target density (evaluated at the mode), and where the degrees of freedom are chosen by the user. Secondly, the mode and scale of g_{mode} are updated using the IS weighted EM algorithm, from equations (30) to equation (36). Note that g_{naive} is already a more advanced candidate than the commonly used g_{mode}; g_{mode} typically yields a numerical efficiency substantially worse than g_{naive}.

(2) **Adaptation**: Estimate the mean and covariance matrix of the target distribution using IS with the draws $\theta^1, \ldots, \theta^N$ from g_{naive}. Use these estimates as the mode and scale matrix of Student's t density $g_{adaptive}$. Draw a sample $\theta^1, \ldots, \theta^N$ from this adaptive Student's t

distribution with density $g_0 = g_{adaptive}$, and compute the IS weights (W^i) for this sample.

(3) **Apply the IS-weighted EM algorithm** given the latest IS weights (W^i) and the drawn sample of step (1). The output consists of the new candidate density g with optimised $\mu_h, \Sigma_h, \nu_h, \eta_h$ for $h = 1, \ldots, H$. Draw a new sample $\theta^1, \ldots, \theta^N$ from the distribution that corresponds to this proposal density and compute the corresponding IS weights (W^i).

(4) **Iterate on the number of mixture components**: Given the current mixture of H components with the corresponding μ_h, Σ_h, ν_h and η_h for $h = 1, \ldots, H$, take x percent of the sample $\theta^1, \ldots, \theta^N$ that correspond to the highest IS weights. Construct a new mode μ_{H+1} with these draws and IS weights and scale matrix Σ_{H+1}, which are the starting values for the additional component in the mixture candidate density. This choice ensures that the new component covers a region of the parameter space in which the previous candidate mixture had relatively too little probability mass. Given the latest IS weights and the sample drawn from the current mixture of H components, apply the IS-weighted EM algorithm to optimize *each* mixture component μ_h, Σ_h, ν_h and η_h with $h = 1, \ldots, H + 1$. Draw a new sample from the mixture of $H + 1$ components and compute the corresponding IS weights.

(5) **Assess convergence of the candidate density quality by inspecting the IS weights** and return to step (3) unless the algorithm has converged.

Cappé et al. (2008) note that there is renewed interest in IS, due to the possibility of parallel processing implementation. Numerical efficiency in sampling methods is related not only to the efficient sample size or relative numerical efficiency but also to the possibility of performing the simulation process in a parallel fashion. Unlike alternative methods, such as the Random Walk Metropolis or the Gibbs Sampler, IS makes use of independent draws from the candidate density, which in turn can be obtained from multi-core CPUs or GPUs. See Durham and Geweke (2011) for a very novel approach. The GPU implementation of MitISEM has been explored in Baştürk et al. (2016).