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On the Existence of Nash Equilibria in Two-Sided Hotelling Models*

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Abstract

The Hotelling model is widely used to analyze platform competition in two-sided markets. In this setup, the degree of platform differentiation must be high relative to cross-group benefits to eschew the alleged non-existence of Nash equilibria. We show instead that Nash equilibria exist even for low differentiation; at such equilibria, platforms avoid competition by replicating a collusive outcome. This result broadens our understanding of the two-sided Hotelling model and is relevant for a wide range of markets − especially digital ones − where platforms operate in low-differentiation environments.

Keywords Hotelling model; Platform competition; Two-sided markets; Equilibrium existence.

JEL Codes: L13, C72, D21.

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1 Introduction

In many markets, competing platforms bridge different groups of agents. Since Armstrong (2006)'s seminal work, the Hotelling duopoly has become the workhorse for analyzing platform competition in such two-sided markets. Crucially, Armstrong (2006) (condition (8) p. 674) shows that platform differentiation must be high relative to the value agents in a group attribute to interacting with agents in the other group (i.e., cross-group benefits) to ensure that Nash price equilibria exist. Otherwise, the lure to steal business from the rival leads to cut-throat competition, which jeopardizes the market's duopolistic structure.[1]

However, in many instances – particularly digital markets like streaming, marketplaces, ride-sharing, and online travel agencies – the assumption of relatively high differentiation is difficult to uphold. On these platforms, finding desired content or services (i.e., cross-group benefits) often matters no less than personal preference for specific platforms (i.e., differentiation degree). One might therefore conclude the Hotelling model is less applicable to digital markets.

In this paper, we consider a simple two-sided Hotelling framework and show that Nash equilibria exist even for low platforms' differentiation. At these equilibria, platforms serve as many consumers as possible, but simultaneously avoid business stealing; such behavior mimics collusion, which defuses ruthless competition. Our result is relevant for two reasons. First, it deepens understanding of the two-sided Hotelling model by characterizing Nash equilibria without restrictive assumptions. Second, it reflects the reality of many digital markets, where low differentiation is a plausible scenario.

In what follows, Section 2 presents the model; to illustrate the Hotelling price game in the unrestricted parameter space, Section 3 considers the one-sided Hotelling model. Section 4 examines the two-sided framework and derives the main result.

2 Model

Two platforms, indexed $i \in \{0,1\}$ and located at the endpoints of a unit-length Hotelling segment, connect two sets of agents, referred to as consumers and producers, and set prices only on the consumer side.[2] The platforms' production costs are normalized to zero.

A unit mass of consumers is uniformly distributed along the line and incurs linear transportation costs. Consumers decide whether to be inactive, or to join one platform. In the former case, their utility is zero; in the latter, the utility function of a consumer located at $x \in [0,1]$

[1] As Caillaud and Jullien (2003) point out, strong cross-group benefits imply "highly contestable market structures, where all potential profits are eroded in order to protect a monopoly position".

[2] We build on Armstrong (2006) for the consumer side and on Hagiu (2006) for the producer side; a similar framework is developed by, e.g., Rasch and Wenzel (2013).

and joining platform i is

$$\mathcal{U}(x, p_i, \mathbb{E}(n_i)) = v + \alpha \cdot \mathbb{E}(n_i) - t d_i(x) - p_i. \tag{1}$$

Term $v \geq 0$ denotes the stand-alone utility that consumers enjoy if they join a platform with no producers. The expression $\alpha \mathbb{E}(n_i)$ captures the cross-group benefits that a consumer gains on platform i, where $\alpha \in (0, 1]$ is the marginal benefit, and $\mathbb{E}(n_i)$ the expected mass of producers active on platform i. The term $t > 0$ is the marginal transportation cost and $d_i(x)$ is the Euclidean distance between the consumer and platform i. Last, p_i is the price charged by platform i on the consumer side.

A unit mass of producers is present on the other side of the market. Each of them chooses whether to be inactive, or to join one or both platforms. Producers bear heterogeneous setup costs f, uniformly distributed over $[0, 1]$, to operate in either platform. Each producer enjoys the marginal cross-group benefit $\gamma \in (0, 1]$ and pays no fee to join a platform. A cost-f producer is willing to join platform i iff $\gamma \cdot \mathbb{E}(D_i) \geq f$, where $\mathbb{E}(D_i)$ is the expected mass of consumers active on platform i. The resulting mass of producers joining platform i is $n_i = Prob(\gamma \cdot \mathbb{E}(D_i) \geq f)$; under the assumption of uniform distribution of setup costs, $n_i = \gamma \cdot \mathbb{E}(D_i)$.

The agents play the following game. Platforms simultaneously set prices p_i to maximize profits, $\pi_i = p_i D_i$. Then, consumers and producers simultaneously make their joining decisions. In the literature, this game is solved as follows. Equation $\mathcal{U}(x, p_0, \mathbb{E}(n_0)) = \mathcal{U}(x, p_1, \mathbb{E}(n_1))$ yields the location of the consumer indifferent between joining platform 0 or 1:

$$\hat{x} = \frac{1}{2} + \frac{p_1 - p_0 + \alpha(\mathbb{E}(n_0) - \mathbb{E}(n_1))}{2t}. \tag{2}$$

Producers and consumers have correct expectations about the other side's participation: $\mathbb{E}(D_0) = \hat{x}$, $\mathbb{E}(D_1) = (1 - \hat{x})$, and $\mathbb{E}(n_i) = \gamma \mathbb{E}(D_i)$. Solving the system of producers' and consumers' expectations yields $\mathbb{E}(n_0) = \gamma \hat{x}$ and $\mathbb{E}(n_1) = \gamma(1 - \hat{x})$, which are then plugged into (2),

$$x_I = \frac{1}{2} - \frac{p_0 - p_1}{2(t - \alpha\gamma)}. \tag{3}$$

Platforms 0 and 1 simultaneously solve problems $\max\limits_{p_0} p_0 x_I$ and $\max\limits_{p_1} p_1(1 - x_I)$. The symmetric equilibrium price is $t - \alpha\gamma$, and the equilibrium profits are $\frac{t-\alpha\gamma}{2}$; both values are positive iff $t > \alpha\gamma$: the transportation cost parameter must be larger than the product of marginal cross-group benefits. This condition is akin to (8) in Armstrong (2006) and is deemed to be necessary and sufficient for equilibrium existence.

In the remainder of the analysis, we proceed in two steps to show that Nash price equilibria exist also if $t \leq \alpha\gamma$. We first describe the Nash equilibria in the one-sided Hotelling framework; next, we re-consider the two-sided framework and derive its Nash equilibria, for *any* $t > 0$.

3 One-sided Framework

Two producers $i \in \{0, 1\}$ are located at the endpoints of the segment and sell their good to the consumers, whose gross utility from unit consumption is v. Letting $t > 0$, at most three alternative price responses are available to i, for any given $p_j \in (0, v]$, $i, j \in \{1, 0\}, i \neq j$.

Producer 0's price responses are depicted in Figure 1, where: p_i is producer i's price; the green (red) line is the consumers' utility when purchasing from 0 (1) as a function of their location x; $x_i(p_i)$ is the (location of) the marginal consumer (indifferent between purchasing from i and not purchasing); $x_I(p_0, p_1)$ is the indifferent consumer; the green (red) portion of the unit segment represents 0's (1's) demand.

Panel (a). Producer 0 sets a relatively low price to steal customers from the rival, so that $x_0(p_0) > x_1(p_1)$: this results into the standard *Hotelling Duopoly* scenario (HD), where the indifferent consumer x_I enjoys positive utility and the market is fully covered.

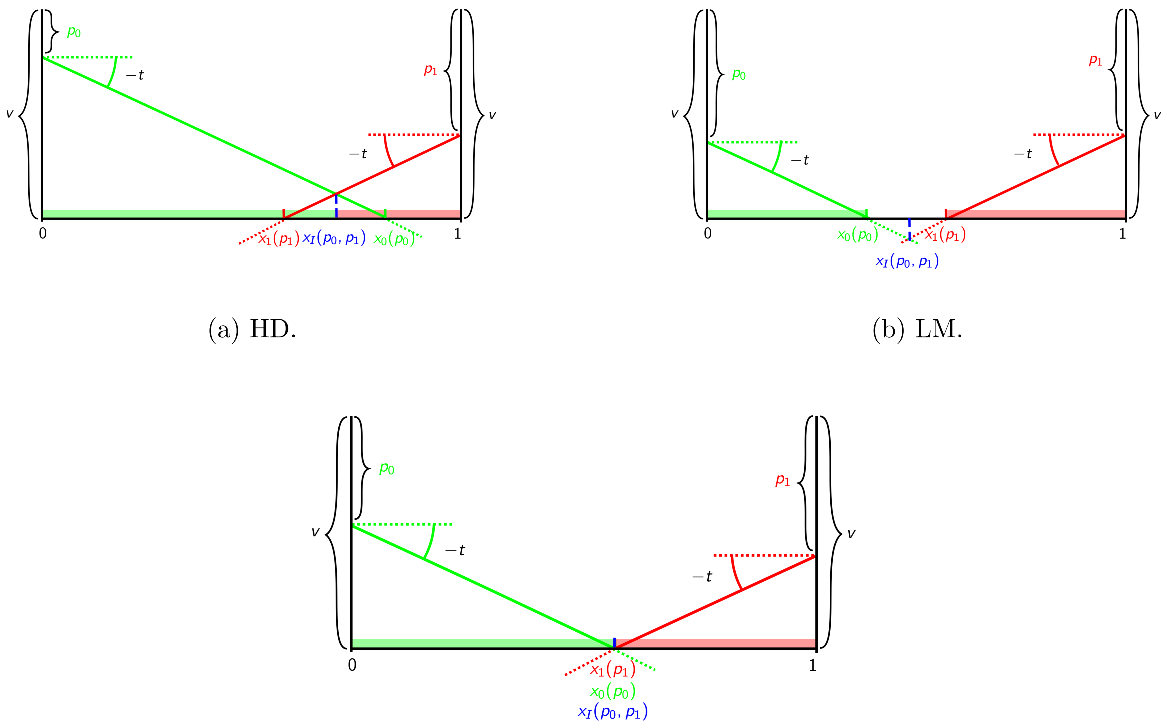
Panel (b). Producer 0 sets a relatively high price to avoid business stealing, so that $x_0(p_0) < x_1(p_1)$: this results into the *Local Monopolies* scenario (LM), where x_I obtains strictly negative utility and the market is partially covered.

Panel (c). Producer 0 sets the price with the aim of *exclusively* attracting the consumers that would derive a negative utility from patronizing producer 1, so that $x_0(p_0) = x_1(p_1)$; x_I obtains zero utility and the market is fully covered. We label this configuration *Monopolistic Duopoly* (MD) because it combines the characterizing features of the other two scenarios: full market coverage and avoidance of business stealing.

Three mutually exclusive Nash Equilibria arise in the one-sided Hotelling price game with unrestricted, positive, t: HD equilibrium if t is low relative v, LM equilibrium if t is high relative v and MD equilibrium if t takes intermediate values.[3]

At the MD equilibrium, producers strategically interact to keep the market covered, hence their prices are interdependent, but do not directly compete with one another to steal consumers. The MD equilibrium prices coincide with those of a multi-product monopolist that serves the entire market. MD is thus akin to, but distinct from, collusion because, on the one hand, the MD prices maximize the industry profits; on the other hand, they are a Nash equilibrium of a one-shot game. We refer to it as *quasi-collusion*.

[3]We refer to Mérel and Sexton (2010) and Thépot (2007) for a characterization. See also: Salop (1979); Cowan and Yin (2008); Rey and Salant (2012); Fedele and Depedri (2016); and Rey and Tirole (2019).



(a) HD.

(b) LM.

(c) MD.

Figure 1: Possible price responses in the one-sided Hotelling game.

4 Results

With the above equilibrium tri-partition in mind, let us now revert to the two-sided framework of Section 2. We define $e \equiv \alpha\gamma$ and state the following.[4]

Proposition 1. *Four alternative equilibrium configurations arise depending on the value of t relative to v and e:*

(i) if $t \leq e$, MD with prices $p^{MD} \equiv v - \frac{t}{2} + \frac{e}{2}$;

(ii) if $e < t < e + \frac{2}{3}v$, HD with prices $p^{HD} \equiv t - e$;

(iii) if $e + \frac{2}{3}v \leq t \leq e + v$, MD with prices $p^{MD} \equiv v - \frac{t}{2} + \frac{e}{2}$;

(iv) if $t > e + v$, LM with prices $p^{LM} \equiv \frac{v}{2}$.

Proof. We first prove points (ii)-(iv) and then point (i).

Point (ii). At the HD equilibrium (see Section 2), the indifferent consumer obtains positive utility, as illustrated in Figure 1: denoting $p^{HD} \equiv t - e$ the equilibrium price, the parametric

[4] We restrict our attention to the unique symmetric MD equilibrium. A continuum of asymmetric MD equilibria exist in the one-sided setup. These equilibria share the same characteristics as the symmetric one, except for the lower total industry profit they generate.

interval in which HD arises is $\mathcal{U}(x_I(p^{HD}), p^{HD}, \mathbb{E}(n_0(p^{HD})) = v + \frac{e}{2} - (t-e) - \frac{t}{2} > 0 \Leftrightarrow (e <)t < e + \frac{2}{3}v$.

Point (iv). Exploiting symmetry, we consider platform 0. Solving $\mathcal{U}(x, p_0, \mathbb{E}(n_0)) = 0$ for x gives the location of the marginal consumer, $x_0 = \frac{v+\alpha\mathbb{E}(n_0)-p_0}{t}$. With correct expectations, platform 0's demand is $D_0 = \frac{v-p_0}{t-e}$ and platform 0 solves problem $\max_{p_0} p_0 \left(\frac{v-p_0}{t-e}\right)$. We get $p^{LM} \equiv \frac{v}{2}$, with $x_0(p^{LM}) = \frac{v}{2(t-e)}$. At the LM equilibrium, the indifferent consumer obtains a negative utility, so $\mathcal{U}(x_I(p^{LM}), p^{LM}, \mathbb{E}(n_0)(p^{LM})) = v + e\frac{v}{2(t-e)} - \frac{v}{2} - \frac{t}{2} < 0 \Leftrightarrow t > e + v$.

Point (iii). Consider the interval $e + \frac{2}{3}v \le t \le e + v$. Substituting (3) into (1) yields the indifferent consumer's utility, $v + (e-t)\left(\frac{1}{2} - \frac{p_0 - p_1}{2(t-e)}\right) - p_0$, which is zero at the MD equilibrium. Hence

$$p_0 + p_1 = 2v - t + e. \tag{4}$$

At the candidate symmetric equilibrium, $p^{MD} \equiv \frac{2v-t+e}{2}$. To confirm that this is an equilibrium, we exploit symmetry and investigate deviations by platform 0, given that $p_1 = p^{MD}$.

Deviation to HD. Plugging $p_1 = (p^{MD} \equiv)\frac{2v-t+e}{2}$ into (2) yields the indifferent consumer's location as a funcion of p_0,

$$\frac{2v+t+e}{4t} - \frac{p_0}{2t} + \frac{\alpha\left(\mathbb{E}(n_0) - \mathbb{E}(n_0)\right)}{2t}. \tag{5}$$

A deviation leads to HD if the deviation price is lower than p^{MD}; in this case, profits are maximized at $p_0^{D,HD} = \frac{2v+t+e+2\alpha(\mathbb{E}(n_0)-\mathbb{E}(n_1))}{4}$. This deviation is unfeasible, iff

$$p_0^{D,HD} \ge p^{MD} \Leftrightarrow \mathbb{E}(n_0) - \mathbb{E}(n_1) \ge \frac{2v - 3t - 3e}{2\alpha}. \tag{6}$$

To investigate whether (6) holds, we proceed as follows. Among all possible consumers' off-the-equilibrium expectations, we select those such that $\mathbb{E}(n_0) - \mathbb{E}(n_1) \ge 0$ and observe that the RHS of (6) is non-positive in the interval of interest. Consequently, reasonable off-equilibrium expectations exist which make this deviation unfeasible.

Deviation to LM. A deviation leads to LM if the deviation price is higher than p^{MD}; in this case, profits are maximized at $p^{LM} \equiv \frac{v}{2}$. However, this value is weakly lower than p^{MD} in the interval of interest, hence this deviation is unfeasible.

Point (i). If $t \le e$, platform i's demand as defined in (3) increases in p_i, which is economically nonsensical. To compute the indifferent consumer's location, we then proceed as follows. At the symmetric MD equilibrium, producers' correct expectations are $\mathbb{E}(D_i) = \frac{1}{2}$, which imply $n_i = \frac{\gamma}{2}$. In turn, consumers' correct expectations require $\mathbb{E}(n_i) = \frac{\gamma}{2}$. Plugging $\frac{\gamma}{2}$ into (1) and solving $\mathcal{U}(x, p_0, \mathbb{E}(n_0)) = \mathcal{U}(x, p_1, \mathbb{E}(n_1))$ for x returns $x_I = \frac{1}{2} + \frac{p_1 - p_0}{2t}$. This indifferent consumer obtains zero utility ($v + \frac{e}{2} - p_0 - t\left(\frac{1}{2} + \frac{p_1-p_0}{2t}\right) = 0$) if (4) holds true. The candidate symmetric MD equilibrium price is hence $p^{MD} \equiv \frac{2v-t+e}{2}$.

Deviation to HD. A deviation to HD is unfeasible iff (6) holds true. With off-equilibrium expectations such that $\mathbb{E}(n_0) - \mathbb{E}(n_1) \geq 0$, it is sufficient to assume that the RHS of (6) is low enough.

Deviation to LM. The deviation price $p^{LM} \equiv \frac{v}{2}$ is lower than p^{MD} in the interval of interest, which rules out this deviation. □

Proposition 1 extends the one-sided Hotelling game equilibrium taxonomy to its two-sided counterpart. Notably, Points (ii)-(iv) parallel those of Proposition 1 in Mérel and Sexton (2010). However, Point (i) highlights a substantive difference: as t approaches 0, the HD equilibrium characterizes one-sided markets, while the MD equilibrium prevails in two-sided environments.

What is the intuition behind this novel result? In two-sided markets with positive cross-group effects, low t would spark a price war if platforms' goal were to steal consumers from one another − as is the case at the HD (candidate) equilibrium − driving prices and profits into negative territory. To avoid this, platforms shift to a MD equilibrium and target only users not served by the rival; this quasi-collusive strategy ensures positive profits.

Our result shows that, contrary to what is commonly accepted in the literature, the Hotelling model is robust enough to describe the equilibrium behavior of platforms when cross-group benefits are at least as important as the differentiation degree; interestingly, this scenario is likely to be common in digital markets.

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